The Curve that Solves the Unsolvable Problems

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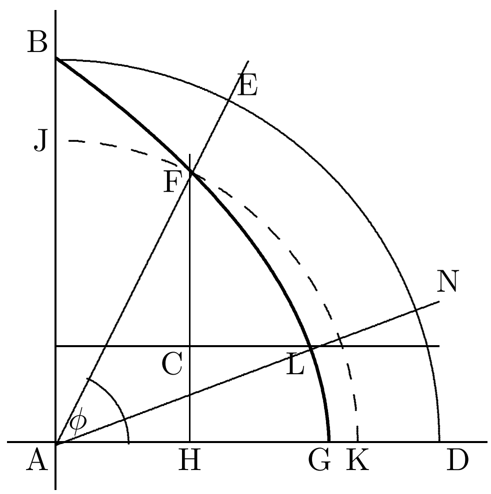
Source: T. Heath, *A History of Greek Mathematics*, Vol. I, Dover, 1981 (originally published 1921).

The trisectrix/quadratrix is attributed to Hippias of Elis (born around 460 BC). The ancient historical manuscripts are ambiguous and inconsistent, but they suggest that Hippias used the curve to trisect the angle and (possibly) some later mathematicians discovered that it could be used to square the circle.

In this drawing, the trisectrix curve (not accurately drawn) is the curve BFLG.



The defining property of the curve can be restated this way: The ratio of the lengths AB and FH equals the ratio of the arcs BED and ED — which is the same as the ratio of the angles BAD (90°) and EAD. In modern notation, if ρ is the length AF and the circle has radius a, then

$$\frac{\rho \sin \phi}{a} = \frac{\phi}{\pi/2}\,.$$

Now suppose that point C is $\frac{1}{3}$ of the way from H to F. Let L be the intersection of the trisectrix with the horizontal line through C, and N be the intersection of the circle with the radial line through L. Then from the defining property of the trisectrix one can see that angles EAD and NAD are in the same ratio as lengths FH and CH, namely 1:3, so EAD has been trisected!

Note that the numerical value of the ratio 1:3 plays no role in the proof. Thus the same curve can be used to construct *any* fraction of a given angle, provided that one is in possession of two line segments in the same proportion!

The squaring of the circle is based on the observation that the arc length BED is to AB as AB is to AG, where G is the bottom point of the curve. Heath, following the Greek sources, takes a page to prove this lemma by contradiction. I shall try to sketch the argument. Suppose that BED:AB :: AB:AK, where K lies to the right of G. Let F be the point where the circle of radius AK intersects the quadratrix, and J the point where it intersects the vertical axis. Then by definition of K, arc JFK equals length AB; but by definition of the quadratrix, AB:FH :: JFK:FK. Therefore, length FH equals arc length FK, which is absurd. A similar argument disposes of the possibility that K lies to the left of G.

The upshot of the lemma is that we can construct two line segments whose lengths are in the ratio BED:AB ($= \pi/2$, in modern language). Therefore, we can construct a rectangle of side lengths a and πa, whose area is that of the circle. Finally, constructing a square with the same area as a given rectangle is an elementary Euclidean construction (see Allen, “Anaxagoras …”, p. 3).