Newton's Divided Difference Interpolation Formula

Source: *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, edited by M. Abramowitz and I. A. Stegun, U.S. Government Priting Office, 1964

They give the formula as

$$f(x) = f_0 + \sum_{k-1}^{n} \pi_{k-1}(x)[x_0, x_1, \ldots, x_k] + R_n$$

$$\begin{array}{llllll}
x_0 & f_0 & & & \\
 & & [x_0, x_1] & & \\
x_1 & f_1 & & [x_0, x_1, x_2] & \\
 & & [x_1, x_2] & & [x_0, x_1, x_2, x_3] \\
x_2 & f_2 & & [x_1, x_2, x_3] & \\
 & & [x_2, x_3] & & \\
x_3 & f_3 & & &
\end{array}$$

$$R_n(x) = \pi_n(x)[x_0, \ldots, x_n, x] = \pi_n(x)\,\frac{f^{(n+1)}(\xi)}{(n+1)!} \qquad ({}_0< \xi < x_n).$$

What does all that mean? Well, the function is assumed to be tabulated at points $x_0 < x_1 < \cdots$, with $f(x_i) = f_i$. Then

$$\pi_n(x) = (x - x_0)(x - x_1)\cdots(x - x_n)$$

is the simplest polynomial that goes exactly through the known points, and the divided differences are

$$[x_0, x_1] = \frac{f_0 - f_1}{x_0 - x_1}\,, \quad \text{etc.,}$$

$$[x_0, x_1, x_2] = \frac{[x_0, x_1] - [x_1, x_2]}{x_0 - x_2}\,,$$

$$[x_0, x_1, \ldots, x_k] = \frac{[x_0, \ldots, x_{k-1}] - [x_1, \ldots, x_k]}{x_0 - x_k}\,,$$

and so on. The triangular pattern above indicates how these numbers are constructed by repeated subtraction. Finally, R_n is the error in the approximation defined by the other terms in the formula.