Lie Groups and Lie Algebras

A Lie group is a group with infinitely many elements that is also a manifold (cf. Chapters 17 and 22). For a quick introduction it is advisable to stick to particular groups, such as the rotations in real n-dimensional space, and to realizations of their elements by matrices. We already started that in the addendum on linear algebra last week, which you might want to review.

Every 2-dimensional rotation can be identified with a 2×2 matrix of the form

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{1}$$

So the group of rotations, SO(2), is a one-dimensional space with the topology of a circle, since $\theta + 2\pi$ represents the same angle as θ.

Another way of looking at (1) is to let θ vary and regard the formula as the parametric equation of a *curve* in the group. (In this case the curve fills up the entire group, but in larger groups the concept of a curve in the group will be less trivial.) Whenever we have parametric equations, we differentiate with respect to the parameter and get a tangent vector! In this case the "vector" is a matrix attached to the curve at each point. In a group, the tangent matrices at different points are related to each other by the group operations, so there is no real loss of information in confining attention to the matrix attached at $\theta = 0$. As demonstrated both in Stillwell Sec. 23.5 and in the linear algebra addendum, when the matrices in the group are orthogonal, the derivative matrices are antisymmetric. For (1) the antisymmetric matrix is

$$R'(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

So far this looks pretty trivial; to get a Lie algebra, we need to generalize the curve a bit. If we replace θ everywhere in (1) by $t\theta$, we have a different parametrized curve, which moves through 0 at a different speed. The corresponding antisymmetric matrix is t times the one in (2). Since t is an arbitrary real number, the (trivial) Lie algebra of all possible tangent vectors to SO(2) at its identity element is isomorphic to the real line. (Note that any other parametrization, say with the angle a nonlinear function of θ, will still have its tangent vector at $\theta = 0$ equal to some multiple of the one in (2), so we haven't left anything out.)

To get a nontrivial example to show what all the fuss is about, we need to go to three dimensions. The group of rotations of Euclidean 3-space is itself a manifold of dimension 3. (As emphasized last week, this agreement of dimensions is an accident!) That is, a rotation requires three angular coordinates to characterize it. The actual parametrized formula is rather complicated and makes some arbitrary choices, just as the coordinatization of a sphere in terms of latitude and longitude requires more than a bit of trigonometry and requires a choice of equator and poles (no problem for the earth, but gratuitous for an abstract perfect sphere). However, the situation simplifies tremendously when one calculates the tangent vector of any curve through the identity: It must be an antisymmetric matrix,

$$R'(0) = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}. \tag{3}$$

And if we vary the speed with which the curve travels through the identity, we can get any 3-vector $\mathbf{v}$ whatsoever. (I should really say "any 3×3 antisymmetric matrix whatsoever," but as explained in the linear algebra addendum, those matrices and the 3-vectors are the same thing, by virtue of the vector cross product. For the quaternionic version of this theory, see Stillwell's exercises for Secs. 23.3–5.) So the Lie algebra of SO(3) is isomorphic, as a vector space, to $\mathbf{R}^3$.

But what makes this a Lie *algebra* is a nontrivial way of multiplying these tangent vectors by each other. The Lie product (or Lie bracket) is not the usual matrix product; it is (for SO(3)) the image of the vector cross product under the identification of vectors with antisymmetric matrices, and it shares the nonassociative property of the vector cross product. In general, the bracket product in a Lie algebra describes how the elements of the original Lie group fail to commute. If U and V are two elements of the group, we can measure their lack of commutativity by examining $(VU)^{-1}(UV) = U^{-1}V^{-1}UV$. You might think that $UV - VU$ would be a more natural thing to look at, but remember that in the abstract Lie group (as opposed to a particular matrix realization of it) *subtraction* is undefined; multiplication and inversion are the only group operations.

On the other hand, the Lie algebra is a vector space; addition and subtraction *are* defined within it. (The Lie algebra is a special case of the *tangent space* defined at any point in a general manifold by considering all the possible directions and speeds at which a curve can pass through that point.) In first-year calculus one learns that the differentiation of a function can be turned around to reconstruct the function "to first order":

$$f(t) \approx f(0) + tf'(0). \tag{4}$$

In third-semester calculus the same thing is done for vector-valued functions: in (4), $f(0)$ becomes the coordinate representation of a point, and $f'(0)$ becomes a vector, tangent to the parametrized curve $f(t)$. Our case is just slightly more sophisticated: f becomes R (as in the previous equations), $R(0) = I$, the identity element of the group, and $R'(0)$ is an element of the Lie algebra. In a matrix representation, I is the identity matrix and $R'(0)$ is (for a rotation group) an antisymmetric matrix, but the first-order Taylor approximation (4) still makes sense (trust me) for the abstract group and Lie algebra, independently of any matrix realization.

So, we can assume that any two group elements close to the identity can be approximated as

$$U = I + tA + O(t^2), \qquad V = I + tB + O(t^2).$$

For the next step I need the second-order approximation, which is not well-defined unless we choose a particular curve. Let's demand that $U(t_1)U(t_2) = U(t_1 + t_2)$, etc., as is true in (1) and the similar parametrizations by angles in SO(3) (see Stillwell p. 511, for example). In that case it can be shown that

$$U = I + tA + \tfrac{1}{2}t^2A^2 + O(t^3), \qquad V = I + tB + \tfrac{1}{2}t^2B^2 + O(t^3)$$

(and, in fact, the entire Taylor series has the form of an exponential series, $U = e^{tA}$.) I will now calculate $U^{-1}V^{-1}UV$ up through order t^2. It is easy to see that $U^{-1} = I - tA + \frac{1}{2}t^2A^2 + O(t^3)$, etc. (This is the fact that a rotation through angle $-t$ is the inverse of the one through t.) Therefore,

$$\begin{aligned}
U^{-1}V^{-1}UV &\sim (I - tA + \tfrac{1}{2}t^2A^2)(I - tB + \tfrac{1}{2}t^2B^2)(I + tA + \tfrac{1}{2}t^2A^2)(I + tB + \tfrac{1}{2}t^2B^2) \\
&= [I - t(A+B) + t^2AB + \tfrac{1}{2}t^2(A^2+B^2)][I + t(A+B)^2 + t^2AB + \tfrac{1}{2}t^2(A^2+B^2] \\
&= I + t^2[-(A+B)^2 + 2AB + A^2 + B^2] \\
&= I + t^2(-A^2 - AB - BA - B^2 + 2AB + A^2 + B^2) \\
&= I + t^2(AB - BA).
\end{aligned}$$

The object $[A, B] = AB - BA$ is called the *Lie bracket*. For the Lie algebra of a matrix group, it can be interpreted literally as the difference of the two matrix products, AB and BA. (For SO(3), it reduces to the vector cross product, as explained last week, or to the quaternion product, as explained in Stillwell. For SO(2) it is trivial, since $[R'(0), R'(0)] = 0$.) For the Lie algebra of a generic Lie group, it is simply a new (nonassociative, anticommutative) product operation defined on the Lie algebra. The Lie bracket, together with the formulas $U = e^{tA}$, tell how to rebuild the group from the algebra in much the same way that a function is recovered from its Taylor series.