Math. 151, WebCalc Sections 507–508 12 December 2001

Final Examination – Solutions

Name: ______________________________ Section: ______________

Part I: Multiple Choice (3 points each)

There is no partial credit. You may not use a calculator.

1. Another word for "perpendicular" is

(A) marginal

(B) differentiable

(C) orthogonal $\Leftarrow$ correct

(D) parametric

(E) asymptotic

2. $\displaystyle\lim_{x\to+\infty} \frac{9x^3 - 4x^2 + 6}{3x^4 + 2x} =$

(A) $+\infty$

(B) $\frac{1}{3}$

(C) $\frac{9}{3}$

(D) $0 \Leftarrow$ correct (x^4 wins over x^3 .)

(E) $\frac{27}{12}$

3. $\displaystyle\lim_{x\to 3^+} \frac{x^2 - x - 6}{x - 3} =$

(A) $5 \Leftarrow$ correct (Factor and cancel, or use l'Hôpital's rule.)

(B) -2

(C) ∞

(D) $-\infty$

(E) Does not exist.

4. $\frac{d}{dx}e^{x^2} =$

(A) e^{x^2-1}

(B) e^{2x}

(C) $2e^x$

(D) $2xe^{2x}$

(E) $2xe^{x^2}$ $\Leftarrow$ correct (chain rule)

5. $\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) =$

(A) 0

(B) $\frac{1}{2}$ $\Leftarrow$ correct (Put over common denominator and use l'Hôpital twice.)

(C) 1

(D) ∞

(E) $-\infty$

6. Suppose that $x^2 + y^3 = 26$. Find $\frac{dy}{dx}$ when $y = 1$ and x is positive.

(A) $\frac{8}{3}$

(B) $-\frac{10}{3}$ $\Leftarrow$ correct (implicit differentiation)

(C) $-\frac{1}{3}$

(D) $\frac{16}{3}$

(E) Can't be found from the information given.

7. Which of these is NOT a valid formula for the derivative of f?

(A) $\lim_{h\to 0} \frac{f(u+h)-f(u)}{h}$ (standard definition)

(B) $f'(1) + \int_1^x f''(t)\,dt$ (fundamental theorem)

(C) $\lim_{h\to 0} \frac{f(x)-f(x-h)}{h}$ (equivalent to standard)

(D) $\lim_{a\to x} \frac{f(x)-f(a)}{a}$ $\Leftarrow$ correct (mixed-up gibberish)

(E) $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ (alternate version of standard)

8. As time t varies from 0 to 1, a particle moves from position $P\,(0,-2)$ to position $Q\,(0,-1)$ along the parametrized curve $\vec{r}(t) = \langle \sin(\pi t), t^2 - 2\rangle$. What is true of the tangent vector (velocity) at the final point Q?

(A) It is equal to $\langle 0, 1\rangle$.

(B) It is parallel to $\langle 0, -1\rangle$.

(C) It is parallel (but not equal) to $\langle 0, 1\rangle$.

(D) It has slope $-\pi$.

(E) It has length $\sqrt{\pi^2+4}$. $\Leftarrow$ correct (Tangent vector is $\langle -\pi, 2\rangle$.)

9. $\lim_{x\to 0} \frac{\sin^2(5x)}{x^2} =$

(A) 0

(B) $\frac{1}{5}$

(C) 1

(D) 5

(E) 25 $\Leftarrow$ correct

10. The left Riemann sum with 4 equal intervals for the integral $\int_1^3 \frac{1}{x}\,dx$ is $\frac{1}{2}\left[1+\frac{1}{1.5}+\frac{1}{2}+\frac{1}{2.5}\right] \approx 1.283$. From this we can conclude that

(A) $\ln 3 = 1.283$.

(B) $\ln 3 < 1.283$ $\Leftarrow$ correct (Left sum is upper sum in this case.)

(C) $\ln 3 > 1.283$

(D) $\ln 3$ is somewhere near 1.283, but it is hard to say whether it is larger or smaller.

(E) None of these.

11. $\frac{d}{dx}\log_{10} x =$

(A) $10\log_9 x$

(B) $\frac{10}{x}$

(C) $\frac{1}{10x}$

(D) $\frac{1}{x\ln 10}$ $\Leftarrow$ correct ($\log_{10} x = \frac{\ln x}{\ln 10}$.)

(E) $\frac{\ln_{10} x}{10}$

12. Radium-226 has a half-life of 1590 years. How long does it take for the number of radium atoms in a sample to decrease to one-third of the original number?

(A) $1590e^{-226/3}$

(B) $\frac{1590}{\ln 3}$

(C) $1590\,\frac{\ln 3}{\ln 2}$ $\Leftarrow$ correct (Use $N(t) = Ce^{-kt}$, $N(1590) = \frac{1}{2}C$, $N(T) = \frac{1}{3}C$.)

(D) $1590^{3/2}$

(E) $226e^{-3180/3}$

13. Use the differential approximation (also known as *linear* or *first-order* approximation) to estimate the amount of paint (in cm^3) needed to apply a coat of paint of thickness h to a hemispherical dome of radius r (both dimensions in centimeters). The volume of a sphere is $V = \frac{4}{3}\pi r^3$.

(A) $2\pi r^2 h$ $\Leftarrow$ correct ($dV = \frac{d}{dr}\left(\frac{2}{3}\pi r^3\right) dr =$ (hemisphere surface area) $\times$ thickness.)

(B) $\frac{2}{3}\pi r^3 h$

(C) $\frac{4}{3}\pi r^2 h$

(D) $\frac{2}{3}\pi r h$

(E) $\frac{2}{3}\pi h^2 r$

14. If $f(a) = 1$, $f'(a) = 3$, and $g(a) = 2$, $g'(a) = -2$, and $h(x) = f(x)/g(x)$, then $h'(a) =$

(A) 4

(B) -4

(C) 2 $\Leftarrow$ correct (quotient rule)

(D) -2

(E) 0

15. $\displaystyle\int_0^{\pi/12} \sin(3x)\, dx =$

(A) $\dfrac{1}{3\sqrt{2}}$

(B) $-\dfrac{1}{3\sqrt{2}} + \dfrac{1}{3}$ $\Leftarrow$ correct (Antiderivative is $-\frac{1}{3}\cos(3x)$.)

(C) $-\dfrac{1}{3\sqrt{2}} - \dfrac{1}{3}$

(D) $-\dfrac{1}{3\sqrt{2}}$

(E) $\dfrac{1}{3\sqrt{2}} - \dfrac{1}{3}$

Part II: Write Out (8 points each)

Show all your work. Appropriate partial credit will be given. You may use a calculator.

16. Evaluate these indefinite integrals.

(a) $\displaystyle\int x\sqrt{x^2+3}\,dx$

Let $u = x^2 + 3$, $du = 2x\,dx$. Then the integral equals

$$\frac{1}{2}\int u^{1/2}\,du = \frac{1}{3}(x^2+3)^{3/2} + C.$$

(b) $\displaystyle\int \cos^4\theta\sin\theta\,d\theta$

Let $u = \cos\theta$, $du = -\sin\theta\,d\theta$. Then the integral equals

$$-\int u^4\,du = -\frac{1}{5}\cos^5\theta + C.$$

17. Gravel is being dumped from a conveyor belt at a rate of 30 ft^3/min. It forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?

The radius of the base is $r = \frac{1}{2}h$. The volume is

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{12}\pi h^3.$$

$$30 = \frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}.$$

$$\frac{dh}{dt} = \frac{120}{\pi}\frac{1}{100} = \frac{6}{5\pi}\text{ ft/min.}$$

18. (a) Evaluate $\int_0^1 \frac{1}{x^2+1}\,dx$.

$$\tan^{-1} x\Big|_0^1 = \tan^{-1} 1 = \frac{\pi}{4}.$$

(b) Calculate $\frac{d}{dx}\sin^{-1}(3x)$.

$$\frac{3}{\sqrt{1-(3x)^2}}$$

19. The manager of a 100-unit apartment complex knows from experience that all units will be occupied if the rent is \$400 per month. A market survey suggests that, on the average, one additional unit will remain vacant for each \$5 increase in rent. What rent should the manager charge to maximize revenue?

Let r be rent and $O(r)$ be the number of units occupied. The model is

$$O(r) = 100 - \left(\frac{r-400}{5}\right) = 180 - \frac{r}{5}.$$

Therefore, the revenue function is

$$R(r) = rO(r) = 180r - \frac{r^2}{5}.$$

Thus at an extremum

$$0 = R'(r) = 180 - \frac{2r}{5},$$

$$r = \$450.$$

This is a maximum because $R'' < 0$.

20. (a) Find all values of the constant b for which the function $y = \sin(bx)$ satisfies the differential equation $y'' + 4y = 0$.

We find $y'' = -b^2\sin(bx)$, so $0 = (-b^2+4)\sin(bx)$. Thus

$$b = \pm 2.$$

(b) Find a solution of $y'' + 4y = 0$ satisfying $y(0) = 1$, $y'(0) = -2$. *Hint:* You will need a cosine term too.

Notice that the solution with $b = -2$ is just the negative of the one with $b = +2$, so there is no need to deal with both. On the other hand, $\cos(2x)$ is another solution, which is essentially different. (The technical term is "linearly independent".) So we can try

$$y = A\cos(2x) + B\sin(2x),$$

where A and B are constants to be found. Note that

$$y' = -2A\sin(2x) + 2B\cos(2x).$$

We need to satisfy

$$1 = A\cos(0) + B\sin(0) = A, \qquad -2 = -2A\sin(0) + 2B\cos(0) = 2B.$$

Thus $A = 1$ and $B = -1$.

$$y = \cos(2x) - \sin(2x).$$

21. Essay question: What is "The Fundamental Theorem of Calculus", and why does it deserve to be called "fundamental"?

[A good answer will state *both* parts of the theorem, will remark that the second part is the primary tool for evaluating definite integrals in practice, and will comment that the two parts of the theorem show how differentiation and integration are related.]

22. Let $f(x) = x^4 - 6x^2$.

(a) Find all relative maxima and minima of f.

We will need the first and second derivatives,

$$f'(x) = 4x^3 - 12x, \qquad f''(x) = 12x^2 - 12.$$

The critical points are at

$$0 = f'(x)/4 = x^3 - 3x = x(x^2 - 3),$$

or $x = 0$ and $\pm\sqrt{3}$. From the first or second derivative test (cf. (c)), we see that $x = 0$ marks a maximum and $x = \pm\sqrt{3}$ mark minima.

(b) Find all inflection points of f.

The candidates are at

$$0 = f'(x)/12 = x^2 - 1,$$

or $x = \pm 1$. Closer inspection (cf. (d)) shows that these are indeed inflection points.

(c) List all intervals where f is **decreasing**.

$$(-\infty, -\sqrt{3}) \quad \text{and} \quad (0, \sqrt{3})$$

(d) List all intervals where f is **concave upward**.

$$(-\infty, -1) \quad \text{and} \quad (1, \infty)$$

(e) Sketch the graph of the function. For full credit on this part, the sketch must be consistent with your answers to the other parts of the question.

[Sorry, I don't have time to typeset graphics or a table. In the process of answering (c) and (d) you should make a table of the signs of f' and f'' in the intervals delimited by the values $x = \pm\infty$, $\pm\sqrt{3}$, ± 1, 0. Calculate

$$f(\pm\sqrt{3}) = -9, \qquad f(\pm 1) = -5, \qquad f(0) = 0,$$

and start your graph by plotting the corresponding points. Another interesting pair of points is at $x = \pm\sqrt{6}$, where the graph crosses the horizontal axis. A good graph will carefully get the concavity correct over each interval, and will not have sharp points at the extrema.]