Theorem 4.1[NC p. 175]:

An arbitrary unitary matrix $U$ can be written in the form

$$U = \begin{bmatrix} e^{i(\alpha - \frac{\beta}{2} - \frac{\delta}{2})} \cos \gamma & -e^{i(\alpha - \frac{\beta}{2} + \frac{\delta}{2})} \sin \frac{\beta}{2} \\ e^{i(\alpha + \frac{\beta}{2} - \frac{\delta}{2})} \sin \frac{\beta}{2} & e^{i(\alpha + \frac{\beta}{2} + \frac{\delta}{2})} \cos \frac{\beta}{2} \end{bmatrix}$$

Proof:

Since an arbitrary complex number $z$ may be written as $z = re^{i\theta}$, for $r \geq 0$ and $\theta \in [0, 2\pi)$, it follows that $U$ may be written as

$$U = \begin{bmatrix} xe^{i\theta_x} & ye^{i\theta_y} \\ ze^{i\theta_z} & we^{i\theta_w} \end{bmatrix},$$

where $x, y, z$ and $w$ are nonnegative real numbers, and $\theta_x, \theta_y, \theta_z, \theta_w \in [0, 2\pi)$.

By the definition of a unitary matrix, $UU^\dagger = I$, so

$$\begin{bmatrix} xe^{i\theta_x} & ye^{i\theta_y} \\ ze^{i\theta_z} & we^{i\theta_w} \end{bmatrix} \begin{bmatrix} xe^{-i\theta_x} & ye^{-i\theta_y} \\ ze^{-i\theta_z} & we^{-i\theta_w} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It follows that $x^2 + y^2 = 1$ and $z^2 + w^2 = 1$, so there exist $\gamma, \psi \in [0, 2\pi)$ such that $x = \cos \gamma, y = \sin \gamma$ and $z = \cos \psi, w = \sin \psi$.

It also follows that $xz e^{i(\theta_x - \theta_z)} + wy e^{i(\theta_y - \theta_w)} = 0$.

Case 1:

Assuming for the moment that $xy \neq 0$, we see that

$$-\frac{wy}{xy} = e^{i(\theta_x - \theta_z - \theta_y + \theta_w)}.$$

Since the left hand side is negative and real, and the right hand side has modulus 1, we see that $\theta_x - \theta_z - \theta_y + \theta_w = (2n + 1)\pi$ for some integer $n$ and $xz = wy$.

Since $xz = wy$, we have $\cos \gamma \cos \psi = \sin \gamma \sin \psi$, so $\cos(\gamma + \psi) = 0$. Therefore, $\gamma + \psi = (2n + 1)\frac{\pi}{2}$, so $z = \cos \psi = \sin \gamma = y$ and $w = \sin \psi = \cos \gamma = x$.

Case 2:

Assume that $x = 0$, the argument for $y = 0$ being similar. We get immediately that $y = 1$ and $w e^{i(\theta_w - \theta_y)} = 0$, so $w = 0$ and $z = 1$
We can solve for $\alpha, \beta$, and $\delta$ by inverting the matrix:

\[
\begin{bmatrix}
1 & -\frac{i}{2} & -\frac{1}{2} \\
1 & +\frac{i}{2} & -\frac{1}{2} \\
1 & -\frac{i}{2} & +\frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\beta \\
\delta
\end{bmatrix}
= 
\begin{bmatrix}
\theta_x \\
\theta_z \\
\theta_w
\end{bmatrix}
\]

to get $\alpha = \frac{1}{2}(\theta_x + \theta_w)$, $\beta = -\theta_x + \theta_z$, and $\delta = -\theta_z + \theta_w$. Rewriting our arbitrary $\gamma$ as $\frac{\gamma}{2}$, the result follows.

**Corollary** [NC p. 20]:

An arbitrary unitary matrix $U$ can be written in the form

\[
e^{i\alpha}
\begin{bmatrix}
e^{-i\beta/2} & 0 & 0 \\
0 & e^{i\beta/2} & 0
\end{bmatrix}
\begin{bmatrix}
\cos \frac{\gamma}{2} & -\sin \frac{\gamma}{2} \\
\sin \frac{\gamma}{2} & \cos \frac{\gamma}{2}
\end{bmatrix}
\begin{bmatrix}
e^{-i\delta/2} & 0 \\
0 & e^{i\delta/2}
\end{bmatrix},
\]

where $\alpha, \beta, \gamma$ and $\delta$ are real.