Some HW problems solved.

p. 79 # 8.

The dependent variable is $y$, since $y$ carries the fleas. Since $y$ is in the denominator, the equation is nonlinear. Hence we apply Theorem 2.4.2.

The equation is already in standard form since the $y'$ is isolated. Hence we note that

$$f(t, y) = (1-t^2-y^2)^{1/2};$$

and

$$\text{diff(rhs(%),y);}$$

must both be real and continuous at the start. That requires that $r^2 + y^2$ be less than one, so the hypothesis is only satisfied at points $(t, y)$ in the interior of the unit circle.

p. 79 # 21

a) There is no solution that passes through the point $(1,1)$.

To receive credit, you must point out the reason why there is none: on p. 75 (19) all solutions are listed. The solution $y = 0$ clearly does not pass through $(1,1)$. To see that no other solution passes through $(1,1)$, we must also rule out other solutions which contain a nonzero value of $y$ of the form $2/3$ for $t$ greater than $t_0$ and $0 \leq t_0$.

Note that such a solution would require $t_0 = -\frac{1}{2}$, which is not allowed.

$$eq:=1=(2/3*(1-2/3)$$

$$solve(%,\{2/3\});$$

b) On the other hand, there is a solution that passes through $(2,1)$.

$$eq:=1=(2/3*(2-2/3)$$

$$solve(%,\{2/3\});$$
c) Look at the graphs shown in the textbook example. The largest positive value of $y$ at $t = 2$ is that given by setting $t_0 = 0$ and $t = 2$, which is the float 1.54. By symmetry and continuity, all values in the interval $(-1.54, 1.54)$ are attained.

\[
eq y = \left(\frac{2}{3} \cdot 2\right)^{\frac{3}{2}}
\]

\[
eq y = \frac{4}{9} \sqrt{4} \sqrt{3}
\]

\[
y = 1.539600718
\]

p. 79 #27.

a) If $n = 0$, then the equation is first order linear, and we apply p. 44 (27).

\[
de := \text{diff}(y(t), t) + p(t) \cdot y(t) = q(t);
\]

\[
\int p(t) \, dt
\]

\[
mu := \text{exp}(\%);
\]

\[
sol := y(t) = 1/mu \cdot (\int mu \cdot q(t), t) + c;
\]

\[
d e := \frac{d}{dt} y(t) + p(t) \cdot y(t) = q(t)
\]

\[
\mu := e^{\int p(t) \, dt}
\]

\[
sol := y(t) = \frac{\int e^{\int p(t) \, dt} q(t) \, dt + c}{e^{\int p(t) \, dt}}
\]

b) If $n = 1$, the the equation is first order linear, and after putting it in standard form, using p. 44 (27) we get

\[
de := \text{diff}(y(t), t) + p(t) \cdot y(t) = q(t) * y(t);
\]

\[
\text{stfrm} := \text{diff}(y(t), t) + (p(t) - q(t)) \cdot y(t) = 0;
\]

\[
\int (p(t) - q(t)) \, dt
\]

\[
mu := \text{exp}(\%);
\]

\[
sol := y(t) = 1/mu \cdot (\int mu \cdot 0, t) + c;
\]

\[
de := \frac{d}{dt} y(t) + p(t) \cdot y(t) = q(t) \cdot y(t)
\]

\[
\text{stfrm} := \frac{d}{dt} y(t) + (p(t) - q(t)) \cdot y(t) = 0
\]
\[
\int (p(t) - q(t)) \, dt \\
\mu := e^{\int (p(t) - q(t)) \, dt}
\]
\[
sol := y(t) = \frac{c}{e^{\int (p(t) - q(t)) \, dt}}
\]

\(c\) Otherwise,

\[\begin{align*}
\text{de} & := \frac{d}{dt} y(t) + p(t) y(t) = q(t) y(t)^n; \\
\text{de} & := \frac{d}{dt} y(t) + p(t) y(t) = q(t) y(t)^n
\end{align*}\]

\[\begin{align*}
\text{subst} & := y(t) = v(t)^{(1/(1-n))}; \\
\text{subs} & (\text{subst}, \text{de}); \\
\text{simplify} & (\%) \\
\text{subst} & := y(t) = v(t)^{\frac{1}{1-n}} \\
\frac{\partial}{\partial t} v(t)^{\frac{1}{1-n}} + p(t) v(t)^{\frac{1}{1-n}} &= q(t) \left( v(t)^{\frac{1}{1-n}} \right)^n \\
-\frac{\frac{\partial}{\partial t} v(t)^{\frac{1}{1-n}}}{1 + n} - p(t) v(t)^{\frac{1}{1-n}} &= q(t) \left( v(t)^{\frac{1}{1-n}} \right)^n \\
-\frac{\frac{\partial}{\partial t} v(t)^{\frac{1}{1-n}}}{1 + n} - p(t) v(t)^{\frac{1}{1-n}} &= q(t) \left( v(t)^{\frac{1}{1-n}} \right)^n
\end{align*}\]

\[\begin{align*}
\text{simplify} & (\%) \\
v(t)^{\frac{-n}{1-n}} \left( v(t)^{-\frac{n}{1+n}} \left( \frac{\partial}{\partial t} v(t)^{\frac{1}{1-n}} \right) - p(t) v(t)^{\frac{-1+n}{1+n}} + p(t) v(t)^{\frac{-1+n}{1+n}} n \right) \\
&= v(t)^{\frac{-n}{1-n}} q(t) \left( v(t)^{\frac{-1+n}{1+n}} \right)^n \\
&- \left( \frac{\partial}{\partial t} v(t)^{\frac{1}{1+n}} \right) v(t) p(t) + v(t) p(t) n \right) \\
&= v(t)^{\frac{-n}{1+n}} q(t) \left( v(t)^{\frac{-1+n}{1+n}} \right)^n
\end{align*}\]

Since the rhs clearly reduces to just \(q(t)\), we get a first order linear equation in \(v(t)\).

p. 79 # 28.

\[\begin{align*}
\text{restart} \\
\text{de} & := 7^2 \left( \frac{d}{dt} y(t) \right)^2 + 2 t y(t) - y(t)^3 = 0; \\
\text{de} & := 7^2 \left( \frac{d}{dt} y(t) \right)^2 + 2 t y(t) - y(t)^3 = 0
\end{align*}\]
We subs and simplify.

```plaintext
> subst:=y(t)=v(t)^(1/(1-3));
  subs(subst,de);
  simplify(%);

\[
\begin{align*}
\text{subst} &:= y(t) = \frac{1}{\sqrt[3]{v(t)}} \\
&\frac{r^2}{2} \left( \frac{\frac{d}{dt} \left( \frac{1}{\sqrt[3]{v(t)}} \right)}{\sqrt[3]{v(t)}} \right) + \frac{2t}{\sqrt[3]{v(t)}} - \frac{1}{v(t)^{3/2}} = 0 \\
&- \frac{1}{2} \frac{r^2}{v(t)^{3/2}} (\frac{d}{dt} v(t) - 4t v(t) + 2) = 0
\end{align*}
\] (12)
```

The first order linear equation in v(t) needs to be put into standard form and the formula on p. 44 (27) applied.

```plaintext
> %*v(t)^(3/2);
  %/(-t^2/2);
  expand(%);
  vde:=%-(2/t^2=2/t^2);

\[
\begin{align*}
&\frac{d}{dt} v(t) - \frac{4v(t)}{t} + \frac{2}{r^2} = 0 \\
&vde := \frac{d}{dt} v(t) - \frac{4v(t)}{t} = -\frac{2}{r^2}
\end{align*}
\] (13)
```

Our integrating factor:

```plaintext
> p:=-4/t;
  int(%,t);
  mu:=exp(%);

\[
p := -\frac{4}{t} \\
-4 \ln(t) \\
\mu := \frac{1}{t^4}
\] (14)
```

p. 44 (27)

```plaintext
> v(t)=1/mu*(int(mu*rhs(vde),t)+c);
  vsol:=expand(%);
```

\[ v(t) = t^4 \left( \frac{2}{5} t^5 + c \right) \]

\[ vsol := v(t) = \frac{2}{5} t + t^4 c \]  \hspace{1cm} (15)

Substitute the formula for \( v(t) \) into the substitution, to get a formula solution for \( y(t) \). Both square roots work.

\[ \text{subs(vsol, subst);} \]
\[ \text{simplify(\%);} \]
\[ y(t) = \frac{1}{\sqrt{\frac{2}{5} t + t^4 c}} \]

\[ \text{ysol := simplify(\%);} \]
\[ y(t) = \frac{\sqrt{5}}{\sqrt{\frac{2}{5} t + t^4 c}} \]
\[ y(t) = \frac{\sqrt{5}}{\sqrt{2 + 5 t^5 c}} \]

\[ ysol := y(t) = \frac{\sqrt{5}}{\sqrt{2 + 5 t^5 c}} \]  \hspace{1cm} (16)

Checking: (A check does not appeal to the back of the book! The back of the book is unavailable on quizzes and exams.)

\[ \text{subs(ysol, de);} \]
\[ \text{simplify(\%);} \]
\[ \rho \left( \frac{\partial}{\partial t} \left( \frac{\sqrt{5}}{\sqrt{2 + 5 t^5 c}} \right) \right) + \frac{2 t \sqrt{5}}{\sqrt{2 + 5 t^5 c}} - \frac{5 \sqrt{5}}{\left( \frac{2 + 5 t^5 c}{t} \right)^{3/2}} = 0 \]
\[ 0 = 0 \]  \hspace{1cm} (17)

p. 79 # 33.

\[ \text{restart:} \]
\[ \text{de1 := diff(y(t), t) + 2*y(t) = 0;} \]
\[ \text{inits1 := y(0) = 1;} \]
\[ del := \frac{d}{dt} y(t) + 2 y(t) = 0 \]
\[ \text{inits1 := y(0) = 1} \]  \hspace{1cm} (18)

Apply p. 44 (27)

\[ \text{p := 2;} \]
\[ \text{int(\%, t);} \]
\[ \mu = \exp(\%) \]
\[ p := 2 \]
\[ 2t \]
\[ \mu := e^{2t} \]  

\[ \text{sol1} := y(t) = 1/\mu * (\int \mu \cdot 0, t) + c; \]
\[ \text{sol1} := y(t) = \frac{c}{e^{2t}} \]

Solve for the constant.

\[ \text{eq1} := 1 = \text{subs}(t=0, \text{rhs}(\%)); \]
\[ \text{csol} := \text{solve}(\%, \{c\}); \]
\[ \text{eq1} := 1 = \frac{c}{e^{0}} \]
\[ \text{csol} := \{c = 1\} \]

Final solution for the interval 0..1,

\[ \text{subs}((\text{csol}, \text{sol1})); \]
\[ \text{sol1} := \text{simplify}(\%); \]
\[ y(t) = \frac{1}{e^{2t}} \]
\[ \text{sol1} := y(t) = e^{-2t} \]

The endpoint of the first half of the solution is the beginning of the second half of the solution.

\[ \text{subs}(t=1, \text{sol1}); \]
\[ y(1) = e^{-2} \]

The differential equation for the second half.

\[ \text{de2} := \text{diff}(y(t), t) + 1 \cdot y(t) = 0; \]
\[ \text{inits2} := \%\%; \]
\[ \text{de2} := \frac{d}{dt} y(t) + y(t) = 0 \]
\[ \text{inits2} := y(1) = e^{-2} \]

Apply p. 44 (27)

\[ \text{p} := 1; \]
\[ \text{int}(\%, t); \]
\[ \mu := \exp(\%); \]
\[ p := 1 \]
\[ t \]
\[ \mu := e^{t} \]
\begin{align}
\text{sol2} := y(t) &= 1/\mu \cdot (\int \mu \cdot 0, t) + c) \\
&= \frac{c}{e^t} \quad (26)
\end{align}

Solve for the constant. Note that the second part of the solution begins where the first part stopped, and that observation tells us how to find the initial value for the second equation.

\begin{align}
\text{eq2} := \text{rhs}(\text{inits2}) &= \text{subs}(t=1, \text{rhs}(\text{sol2})) \\
\text{csol} := \text{solve}(%, \{c\}) \\
&= e^{-2} = \frac{c}{e} \\
&= \{ c = e^{-2} e \} \quad (27)
\end{align}

The final solution for the second half.

\begin{align}
\text{subs}(\text{csol}, \text{rhs}(\text{sol2})) \\
\text{sol2} := y(t) &= \text{simplify}(%) \\
&= \frac{e^{-2} e}{e^t} \\
&= y(t) = e^{-t - 1} \quad (28)
\end{align}

Put the two halves together in a piecewise function:

\begin{align}
\text{sol} := y(t) &= \text{piecewise}(0 < t \text{ and } t \leq 1, \text{rhs(sol1)}, t > 1, \text{rhs(sol2)}) \\
&= \begin{cases}
  e^{-2 t} & 0 < t \text{ and } t \leq 1 \\
  e^{-t - 1} & 1 < t
\end{cases} \quad (29)
\end{align}

Plot the solution:

\begin{align}
\text{plot(} \text{rhs(sol)}, t=0..3); 
\end{align}