This is a second order linear homogeneous constant coefficient differential equation. The WAG is $y(t) = e^{rt}$.

\[
\begin{align*}
\text{restart:} & \\
\text{assume(t,real):} & \\
\text{de:=} &\text{diff(y(t),t$2$)} - 2*\text{diff(y(t),t)} + 2*y(t) = 0; \\
\text{sol:=} &\text{y(t)=exp(r*t);} \\
\text{de} &:= \frac{d^2}{dr^2} y(t) - 2 \left( \frac{d}{dr} y(t) \right) + 2 y(t) = 0 \\
\text{sol} &:= y(t) = e^{rt} \\
\end{align*}
\]

(1)

Note that the eigenvalues come as complex conjugate pairs.

\[
\begin{align*}
\text{subs(sol,de);} & \\
\text{simplify(%)}; & \\
\text{char_eq:=} &\frac{%}{\text{exp(r*t)};} \\
\text{rsol:=} &\text{solve(%,r);} \\
\text{rsol[1]}; & \\
\text{exp(%)t}; & \\
\text{complex_sol:=} &\text{evalc(%)}; \\
\text{complex_sol} &:= e^{(1+1)t} \\
\end{align*}
\]

(2)

\[
\begin{align*}
\text{sol1:=} &\text{Re(complex_sol);} \\
\text{sol2:=} &\text{Im(complex_sol);} \\
\text{FS:=} &\text{[sol1,sol2]}; \\
\text{sol1} &:= e^t \cos(t) \\
\text{sol2} &:= e^t \sin(t) \\
\text{FS} &:= [e^t \cos(t), e^t \sin(t)] \\
\end{align*}
\]

(4)

Since the solutions are complex conjugate pairs, then the real and imaginary parts of the complex answers are also solutions, by the Superposition Theorem.

We check that the real solutions that we got are linearly independent by checking that the Wronskian determinant is different than zero.
Since the equation is linear and homogeneous, and since the two real solutions are linearly independent, we can apply the Superposition Theorem to get all solutions, i.e., the general solution.

If we convert the second order differential equation into a system of two first order differential equations, we begin with new variables.

\[ z_1(t) = y(t) \]
\[ z_2(t) = y'(t) \]

The differential equations for the \( z \) variables are then

\[ z_1'(t) = z_2(t) \]
\[ z_2'(t) = -2z_1(t) + 2z_2(t) \]

Note that the solution vector \( Z(t) \) is then

\[ Z := \begin{bmatrix} c_1 e^t \cos(t) + c_2 e^t \sin(t) \\ c_1 e^t \cos(t) - c_1 e^t \sin(t) + c_2 e^t \sin(t) + c_2 e^t \cos(t) \end{bmatrix} \]
The trajectories spiral exponentially out from the origin. While the origin is an equilibrium point, meaning that if we are at the origin the derivatives are zero so we don't move, if we get just a little bit off the origin, the arrows push us away and we never come back. The equilibrium is unstable.

Note how the lower trajectory crosses the x-axis about 3/2 and crosses the y-axis about -5.

We can also get the same result by using a parametric plot. Let's go back in and take the first point plotted, \( z_1(0)=1, z_2(0)=2 \).

\[
> \text{with(DEtools):} \\
> \text{DEplot([de1, de2], [z1(t), z2(t)], t=-3..3, [[z1(0)=1, z2(0)=1], [z1(0)=2, z2(0)=2]], z1=-7..7, z2=-7..7, linecolor=black)};
\]

We can solve this easily by inspection, getting \( c_1=1, c_2=0 \). Now let's put those values of \( c \) into the \( Z \) vector and do a parametric plot.
\[ Z_{\text{vec}} := \begin{bmatrix} e^t \cos(t) \\ e^t \cos(t) - e^t \sin(t) \end{bmatrix} \]

We plot \( z_1 \) as the x-coordinate, and \( z_2 \) as the y-coordinate. Again, the trajectory crosses the x-axis about 3/2 and crosses the y-axis about -5.

\[ \text{xc} := Z_{\text{vec}}[1,1]; \]
\[ \text{yc} := Z_{\text{vec}}[2,1]; \]
\[ \text{plot}([\text{xc, yc, t=-3..2}, \text{scaling=constrained}]); \]

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The dependent variable is \( y \). The equation is linear, homogeneous, and Cauchy-Euler. The WAG is \( y(x) = x^r \).
To apply the EU Theorem to this differential equation, it must first be put in standard form with lead coefficient 1, so we need to divide by $x^2$. After we have divided, the functions that must be continuous at the start are $-\frac{4}{x}$, $-\frac{6}{x^2}$, and 0. There as a discontinuity at $x = 0$, so to start we must either begin and stay to the left of $x = 0$ or begin and stay to the right of $x = 0$. The book selected to the right of $x = 0$, i.e., $x > 0$.

To apply the EU Theorem to this differential equation, it must first be put in standard form with lead coefficient 1, so we need to divide by $x^2$. After we have divided, the functions that must be continuous at the start are $-\frac{4}{x}$, $-\frac{6}{x^2}$, and 0. There as a discontinuity at $x = 0$, so to start we must either begin and stay to the left of $x = 0$ or begin and stay to the right of $x = 0$. The book selected to the right of $x = 0$, i.e., $x > 0$.

```latex
de := x^2 \cdot \frac{d^2}{dx^2} y(x) + 3x \cdot \frac{d}{dx} y(x) - 6y(x) = 0
sol := y(x) = x^r
```

We subs and simplify the explicit solution to get the indicial equation, from which we find the values of $r$.

```latex
\text{subs(sol,de)};
\text{simplify(\%)};
\text{Ind_eqn := \%/x^r;}
\begin{align*}
x^2 \left( \frac{d^2}{dx^2} x^r \right) + 3x \left( \frac{d}{dx} x^r \right) - 6x^r &= 0 \\
x^r \left( r^2 + 2r - 6 \right) &= 0 \\
\text{Ind_eqn} := r^2 + 2r - 6 &= 0
\end{align*}
\tag{12}
```

We find two solutions using the values of $r$.

```latex
\text{rsol := solve(\%, r);}
\begin{align*}
\text{rsol} &= -1 + \sqrt{7}, -1 - \sqrt{7}
\end{align*}
\tag{13}
```

The two solutions are linearly independent, since the Wronskian determinant is different than zero.

```latex
\text{with(VectorCalculus);}\quad \text{with(LinearAlgebra);}\quad \text{with(VectorCalculus);}\quad \text{with(LinearAlgebra)};
\text{FS := [sol1, sol2];}
\text{W := Wronskian(FS,x);}\quad \text{Determinant(\%);}\quad \text{simplify(\%);}\quad \text{simplify(\%)};
\begin{align*}
\text{FS} &= \left[ x^{-1 + \sqrt{7}}, x^{-1 - \sqrt{7}} \right]
\end{align*}
```
Since the equation is linear and homogeneous, we can apply the Superposition Theorem to get infinitely many answers by taking linear combinations of answers. Since the two solutions are linearly independent, we get all solutions, and the general solution is

\[ \text{gen_sol} := c_1 \text{sol1} + c_2 \text{sol2} ; \]

\[ \text{gen_sol} := c_1 x^{-1+\sqrt{7}} + c_2 x^{-1-\sqrt{7}} \]