In this problem, the authors have computed the eigenvalues and eigenvectors and presented us with two linearly independent solutions. Hence all that we need to do is to assemble them into a fundamental solution matrix and apply Theorem 4.8.1.

```maple
restart:
interface(labeling=false):
with(LinearAlgebra):
A:=Matrix(<<-1,0>|<-2,1>>);
g:=Matrix(<exp(-t),t>);

A := [ -1, -2 ]
     [  0,  1 ]

(1)

Assemble the given solutions in a fundamental solution matrix and check that what we made is fundamental.

s1:=exp(-t)*Matrix(<1,0>);
s2:=exp(t)*Matrix(<-1,1>);

s1 := [ e^t ]
     [  0  ]

s2 := [ -e^t ]
     [  e^t ]

(2)

Assemble the matrix.

F:=Matrix(<s1|s2>);

F := [ e^t, -e^t ]
     [  0,  e^t ]

(3)

Check two things: 1) the columns are homogeneous solutions and 2) The columns are linearly independent.

Check of 1).

map(diff,F,t)-A.F;

(4)
Check of 2).

\[
\begin{align*}
\text{Determinant}(F) &; \\
\text{simplify}(&); \\
& \frac{e^{-t}e^t}{1} \\
\end{align*}
\]

Now we use the fundamental solution matrix in Theorem 4.8.1. The first two Maple commands do the integral in (15), the third Maple command adds the constants of integration, the fourth Maple command multiplies the integral by the fundamental matrix in (15). The last Maple command simplifies all the entries in the matrix to make them pretty.

\[
\begin{align*}
> & (1/F).g; \\
& \text{map(int,} &%;t); \\
& %+\text{Matrix}(c1,c2); \\
& F.%; \\
& \text{gen_sol:=map(simplify} &%); \\
\end{align*}
\]

\[
\begin{vmatrix}
1 + \frac{t}{e^{-t}}
\end{vmatrix}
\]

\[
\begin{vmatrix}
\frac{t}{e^t}
\end{vmatrix}
\]

\[
\begin{vmatrix}
t + \frac{t}{e^{-t}} - \frac{1}{e^{-t}}
\end{vmatrix}
\]

\[
\begin{vmatrix}
- \frac{1+t}{e^t}
\end{vmatrix}
\]

\[
\begin{vmatrix}
t + \frac{t}{e^{-t}} - \frac{1}{e^{-t}} + cl
\end{vmatrix}
\]

\[
\begin{vmatrix}
- \frac{1+t}{e^t} + c2
\end{vmatrix}
\]

\[
\begin{vmatrix}
e^{-t} \left( t + \frac{t}{e^{-t}} - \frac{1}{e^{-t}} + cl \right) - e^t \left( - \frac{1+t}{e^t} + c2 \right)
\end{vmatrix}
\]

\[
\begin{vmatrix}
e^t \left( - \frac{1+t}{e^t} + c2 \right)
\end{vmatrix}
\]
\begin{align*}
\text{gen\_sol} := \begin{bmatrix}
e^{-t} t + 2 t + e^{-t} c1 - e^{-t} c2 \\
-e^{-t} (e^{-t} + e^{-t} t - 2c1)
\end{bmatrix}
\end{align*}

Check to see that the solution works: we are supposed to have \( \text{sol}' = A \cdot \text{sol} + g \), so \( \text{sol}' - A \cdot \text{sol} - g \) should be zero.

\begin{align*}
\text{map}(\text{diff}, \text{gen\_sol}, t) - A \cdot \text{gen\_sol} - g; \\
\text{map}(\text{simplify}, \%) \\
\begin{bmatrix}
2 - 2 e^{t} c2 + 2 t - 2 e^{t} (e^{-t} + e^{-t} t - c2) \\
e^{t} e^{t} t - t \\
0 \\
0
\end{bmatrix}
\end{align*}

p. 289 #7.

The EU Theorem guarantees us that there should be one and only one solution. We set up two equations to solve for the two values of \( c \), and get the unique solution.

When \( t = 0 \), the entry in entry in the first row and first column of \( \text{gen\_sol} \) should be -1.

\begin{align*}
> -1 = \text{subs}(t=0, \text{gen\_sol}[1,1]); \\
\text{eq1} := \text{simplify}(\%); \\
-1 = e^{0} c1 - e^{0} c2 \\
\text{eq1} := -1 = c1 - c2
\end{align*}

When \( t = 0 \), the entry in the second row and first column of \( \text{gen\_sol} \) should be 1.

\begin{align*}
> 1 = \text{subs}(t=0, \text{gen\_sol}[2,1]); \\
\text{eq2} := \text{simplify}(\%); \\
1 = -e^{0} (e^{0} - c2) \\
\text{eq2} := 1 = -1 + c2
\end{align*}

Solve for the values of \( c \), then substitute into \( \text{gen\_sol} \) to get the unique solution that matches the initial values.

\begin{align*}
> \text{csol} := \text{solve}\{\text{eq1, eq2}\}, \{c1, c2\}; \\
\text{unique} := \text{subs}(\text{csol}, \text{gen\_sol}); \\
\text{csol} := \{c1 = 1, c2 = 2\} \\
\text{unique} := \begin{bmatrix}
e^{-t} t + 2 t + e^{-t} - 2 e^{t} \\
-e^{t} (e^{-t} + e^{-t} t - 2)
\end{bmatrix}
\end{align*}

Check that the unique solution matches the initial values:
The equation is neither constant coefficient nor Cauchy-Euler, so we cannot find homogeneous solutions. However, we do not need to find homogeneous solutions, since the problem gives them to us. Our job is to verify that the solutions are correct and that we have a fundamental solution set, then apply the matrix inverse, map int, dot product to find a particular solution by variation of parameters.

The EU Theorem requires the differential equation to be in standard form, so we need to divide by the lead coefficient to find a good place to start. Since we cannot start at \( t = 0 \), we have to start to the right of zero or to the left of zero, and the problem chose to start to the right.

```plaintext
> restart:
interface(labeling=false):
de:=t*diff(y(t),t$2)-(1+t)*diff(y(t),t)+y(t)=t^2*exp(2*t);
hde:=lhs(de)=0;

\[
de := t \left( \frac{d^2}{dt^2} y(t) \right) - (1 + t) \left( \frac{d}{dt} y(t) \right) + y(t) = t^2 e^{2t}
\]

\[
hde := t \left( \frac{d^2}{dt^2} y(t) \right) - (1 + t) \left( \frac{d}{dt} y(t) \right) + y(t) = 0
\]

Check that the explicit solutions form a fundamental solution set.

```plaintext
> sol1:=(1+t);
sol2:=exp(t);

\[
\text{sol1} := 1 + t \\
\text{sol2} := e^t
\]

Check solutions:

```plaintext
> hsol:=y(t)=c1*sol1+c2*sol2;

\[
hsol := y(t) = c_1 (1 + t) + c_2 e^t
\]

Check that solutions are linearly independent:

```plaintext
> with(VectorCalculus):

\[
t \left( \frac{\partial^2}{\partial t^2} (c_1 (1 + t) + c_2 e^t) \right) - (1 + t) \left( \frac{\partial}{\partial t} (c_1 (1 + t) + c_2 e^t) \right) + c_1 (1 + t) + c_2 e^t = 0
\]

\[
0 = 0
\]
with(LinearAlgebra):
FS:=[sol1, sol2];
W:=Wronskian(FS,t);
Determinant(%);

\[
FS := \begin{bmatrix} 1 + t, e^t \end{bmatrix}
\]
\[
W := \begin{bmatrix} 1 + t & e^t \\
1 & e^t \\
e^t \end{bmatrix}
\]

Now find a particular solution using the matrix inverse, map int, dot product method of variation of parameters.

> R:=Matrix(<0, t^2*exp(2*t)/t>);

\[
R := \begin{bmatrix} 0 \\
t e^{2t} \end{bmatrix}
\]

Find the derivatives of the v's.

> Vp:=1/W.R;

\[
Vp := \begin{bmatrix} -e^{2t} \\
\frac{1 + t}{e^t} \frac{e^{2t}}{e^t} \end{bmatrix}
\]

Integrate to find the v's.

> V:=map(int, %, t);

\[
V := \begin{bmatrix} -\frac{1}{2} e^{2t} \\
t e^{2t} \\
\frac{e^t}{e^t} \end{bmatrix}
\]

Create a particular inhomogeneous solution using the dot product.

> convert(V, Vector) . convert(FS, Vector);

\[
psol := y(t) = \text{simplify}(\%);\]
\[
-\frac{1}{2} e^{2t} (1 + t) + t e^{2t}
\]

Add the homogeneous solutions to get the general solution:
gen_sol := y(t) = \frac{1}{2} e^{2t} (-1 + t) + cl (1 + t) + c2 e^t \quad (20)

Check the explicit gen_sol to see that it works.

\[ \text{subs(gen_sol, de); simplify(\%); } \]
\[ l \left( \frac{\partial^2}{\partial t^2} \left( \frac{1}{2} e^{2t} (-1 + t) + cl (1 + t) + c2 e^t \right) \right) - (1 + t) \left( \frac{\partial}{\partial t} \left( \frac{1}{2} e^{2t} (-1 + t) + cl (1 + t) + c2 e^t \right) \right) + \frac{1}{2} e^{2t} (-1 + t) + cl (1 + t) + c2 e^t = t^2 e^{2t} \]
\[ t^2 e^{2t} = t^2 e^{2t} \quad (21) \]

p. 290 # 29.

\[ \text{restart: de := x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+4*y(x)=x^2*ln(x); } \]
\[ \text{hde := lhs(de)=0; } \]
\[ \text{de := x^2 \left( \frac{d^2}{dx^2} y(x) \right) - 3 x \left( \frac{d}{dx} y(x) \right) + 4 y(x) = x^2 \ln(x) } \]
\[ \text{hde := x^2 \left( \frac{d^2}{dx^2} y(x) \right) - 3 x \left( \frac{d}{dx} y(x) \right) + 4 y(x) = 0 } \quad (22) \]

The WAG for Cauchy-Euler is \( y(x) = x^r \). Subs and simplify the explicit solution to find the indicial equation and find the eigenvalues.

\[ \text{hsol := y(x)=x^r; } \]
\[ \text{subs(hsol, hde); simplify(\%); } \]
\[ \%/x^r; \]
\[ \text{hsol := y(x) = x^r } \]
\[ x^r (r^2 - 4r + 4) = 0 \]
\[ r^2 - 4r + 4 = 0 \quad (23) \]

We have repeated eigenvalues for the Cauchy-Euler equation.

\[ \text{rsol := solve(\%, r); } \]
\[ \text{rsol := 2, 2 } \quad (24) \]
\[ \text{soll := x^rsol[1]; sol2 := soll*ln(x); } \]
\[ \text{soll := x^2 } \]
\[ \text{sol2 := x^2 \ln(x) } \quad (25) \]
Check that we have a fundamental solution set:

Check solutions:

\[
hsol := y(x) = c_1 x^2 + c_2 x^2 \ln(x)
\]

\[
x^2 \left( \frac{\partial^2}{\partial x^2} \left( c_1 x^2 + c_2 x^2 \ln(x) \right) \right) - 3 x \left( \frac{\partial}{\partial x} \left( c_1 x^2 + c_2 x^2 \ln(x) \right) \right) + 4 c_1 x^2 + 4 c_2 x^2 \ln(x) = 0
\]

0 = 0

Check that the solutions are linearly independent.

\[
\text{with(VectorCalculus):}
\text{with(LinearAlgebra):}
FS := \{sol1, sol2\};
W := Wronskian(FS, x);
\text{Determinant(\%)};
\]

\[
FS := [x^2, x^2 \ln(x)]
\]

\[
W := \begin{bmatrix}
x^2 & x^2 \ln(x) \\
2 x & 2 x \ln(x) + x \\
x^3 & 
\end{bmatrix}
\]

Use the matrix inverse, map int, dot product to find a particular solution by variation of parameters.

\[
R := \text{Matrix}(<0, x^2 \ln(x) / x^2>);
\]

\[
R := \begin{bmatrix}
0 \\
\ln(x)
\end{bmatrix}
\]

Find the derivatives of the v's:

\[
Vp := 1/W.R;
\]

\[
Vp := \begin{bmatrix}
- \frac{\ln(x)^2}{x} \\
\frac{\ln(x)}{x}
\end{bmatrix}
\]

Integrate to find the v's:

\[
V := \text{map(int, Vp, x)};
\]
Add the homogeneous solutions to our particular solution to find a general solution.

\[ \text{psol} := y(x) = \frac{1}{6} \ln(x)^3 x^2 \]

Add the homogeneous solutions to our particular solution to find a general solution.

\[ \text{gen_sol} := y(x) = \frac{1}{6} \ln(x)^3 x^2 + c1 x^2 + c2 x^2 \ln(x) \]