To have a Laplace transform, the function must be of exponential order. To have \( \frac{e^{-4t}}{e^{st}} \) go quickly to zero, we must have \(-4 < s\), producing a polynomial divided by an exponential. We use our assumption that \( s > -4 \) to make the improper Laplace integral converge.

\[
\begin{align*}
\text{assume}(s > -4); & \quad \int \exp(-4t) (1+t^2)^2 \exp(-s t) , t = 0..\infty; \\
& \quad \frac{344 + 100 s^2 + 288 s + s^4 + 16 s^3}{(4 + s) \left(16 + 8 s + s^2\right)^2} \\
\end{align*}
\]

We can also take the Laplace transform by multiplying out the square of the binomial, using linearity, and using the table on p. 325.

\[
\begin{align*}
> (t^2+1)^2; & \quad \text{expand(\%);} \\
& \quad \exp(-4t) \times \%; \\
& \quad \frac{(r^2 + 1)^2}{e^{-4t} (r^4 + 2 r^2 + 1)} \\
\end{align*}
\]

Using line 2 of the table, we see that the Laplace transform of \( e^{-4t} \) is \( \frac{1}{s + 4} \). Using line 11, we see that the Laplace transform of \( r^2 e^{-4t} \) is \( \frac{2!}{(s + 4)^3} \). Using line 11 again, we see that the Laplace transform of \( r^4 e^{-4t} \) is \( \frac{4!}{(s + 4)^5} \). Now using linearity, we get the Laplace transform is the same that we got above.

\[
\begin{align*}
> 1/(s+4) + 2*2!/(s+4)^3 + 4!/(s+4)^5; & \quad \text{simplify(\%);} \\
& \quad \frac{24}{(4 + s)^5} + \frac{4}{(4 + s)^3} + \frac{1}{4 + s} \\
& \quad \frac{344 + 100 s^2 + 288 s + s^4 + 16 s^3}{(4 + s)^5} \\
\end{align*}
\]

We can also compute the Laplace transform using the integral transform package, inttrans, and the laplace command.

\[
\begin{align*}
> \text{with(inttrans);} & \quad \exp(-4t) \times (t^2+1)^2; \\
\end{align*}
\]
One can memorize Corollary 5.2.3, or simply rederive the result from using 5.2.2 twice in a row:

\[
L(f'') = L((f')') = sL(f') - f'(0) = s(sL(f) - f(0)) - f'(0) = s^2L(f) - sf(0) - f'(0).
\]

Maple already knows the table, so we can simply give the laplace command. Note that \(D(f)(0)\) is Maple's way of writing \(f'(0)\).

```maple
> restart:
with(inttrans):
diff(f(t),t$2);
laplace(%,t,s);
```

One can make the answer come out in prettier form by using an alias and our convention that the Laplace transform of \(f(t)\) is \(F(s)\), but that alias is not required.

```maple
> alias(F(s)=laplace(f(t),t,s)):
diff(f(t),t$2);
laplace(%,t,s);
```

As a matter of fact, we can take the Laplace transform of the whole equation quickly and efficiently using the Laplace command.

As you look at the Maple output, observe that the first three terms come from Corollary 5.2.3 applied to \(9 \left( \frac{d^2}{dt^2} y(t) \right)\), the next two terms come from Corollary 5.2.3 applied to \(12 \left( \frac{d}{dt} y(t) \right)\), and the last term is just the Laplace transform of \(y(t)\), using superposition.

```maple
> alias(Y(s)=laplace(y(t),t,s)):
de:=9*diff(y(t),t$2)+12*diff(y(t),t)+4*y(t)=0;
laplace(de,t,s);
```
As a matter of fact, we can complete the process of including the initial value \( y(0) = 2, y'(0) = -1 \) by using the Maple `subs` command. We then take all the terms that do not have \( Y(s) \) and move them to the rhs, getting \( 9 s^2 Y(s) + 12 s Y(s) + 4 Y(s) = 18 s + 15 \). Factor out the \( Y(s) \) to get \( 9 s^2 + 12 s + 4 \) \( Y(s) = 18 s + 15 \). We can then solve for \( Y(s) \) by dividing.

\[
Y(s) = \frac{18 s + 15}{9 s^2 + 12 s + 4}
\]

\[\text{alias}(Y(s) = \text{laplace}(y(t), t, s)):\]
\[\text{de} := 9 \frac{d^2}{dt^2} y(t) + 12 \frac{dy}{dt} y(t) + 4 y(t) = 0;\]
\[\text{inits} := y(0) = 2, D(y)(0) = -1;\]
\[\text{laplace}(\text{de}, t, s);\]
\[\text{subs}(\text{inits}, \%);\]
\[Y(s) = \text{solve}(\%, Y(s));\]

\[
de := 9 \frac{d^2}{dt^2} y(t) + 12 \frac{dy}{dt} y(t) + 4 y(t) = 0
\]
\[\text{inits} := y(0) = 2, D(y)(0) = -1\]
\[9 s^2 Y(s) - 9 D(y)(0) - 9 s y(0) + 12 s Y(s) - 12 y(0) + 4 Y(s) = 0\]
\[9 s^2 Y(s) - 15 - 18 s + 12 s Y(s) + 4 Y(s) = 0\]
\[Y(s) = \frac{3 (5 + 6 s)}{9 s^2 + 12 s + 4}\]

p. 322 #25.

To take the Laplace of the equation, we can apply p. 318 (4) to get the Laplace of the left hand side, and break up the Laplace integral of the function on the right, as we did on p. 315 # 14.

As an alternative, we can write the function as a piecewise function and use Maple’s `laplace` command.

\[
\text{de} := \frac{d^2}{dt^2} y(t) + 4 y(t) = \begin{cases} t & 0 \leq t \text{ and } t < 1 \\ 1 & 1 \leq t \text{ and } t < \infty \end{cases}
\]
\[\text{inits} := y(0) = 0, D(y)(0) = 0;\]

Since the initial values are both zero, the second and third terms disappear.

\[
\text{laplace}(\text{de}, t, s);\]
\[\text{subs}(\text{inits}, \%);\]
\[ s^2 Y(s) - D(y)(0) - sy(0) + 4Y(s) = \frac{1-e^{-s}}{s^2} \]
\[ s^2 Y(s) + 4Y(s) = \frac{1-e^{-s}}{s^2} \]  \hspace{1cm} (10)

As in the previous problem, we can then factor out \( Y(s) \) and divide both sides by \( s^2 + 4 \).

\[
Y(s) = \text{solve}(%, Y(s));
\]
\[
Y(s) = -\frac{1 + e^{-s}}{s^2 (s^2 + 4)}
\]  \hspace{1cm} (11)

p. 322 # 26.

We set up the mixture problem exactly as we did in the modeling problems Section 2.3.

Let \( q(t) \) be the mass in lbs of the salt in the tank at time \( t \) minutes. The rate in is 2, the concentration in is 1/2. The rate out is 2, so the volume stays constant at 100, and the concentration out is \( \frac{q(t)}{100} \). Since at the start, there is only fresh water, \( q(0) = 0 \).

\[
\text{restart:}
\]
\[
\text{interface(labeling=false):}
\]
\[
\text{with(inttrans):}
\]
\[
de := \text{diff}(q(t),t) = 2 \times \frac{1}{2} - 2q(t)/100;
\]
\[
inits := q(0) = 0;
\]
\[
de := \dot{q}(t) = 1 - \frac{q(t)}{50}
\]
\[
inits := q(0) = 0
\]  \hspace{1cm} (12)

Here is a "classical" solution, similar to the solution given to HW p. 80 #33, which you should now re-examine.

Since this is a modeling problem, we could solve using dsolve, to get a solution which is valid for the first ten minutes, which we now do:

\[
\text{sol := dsolve}\{\{de, inits\}, q(t)\};
\]
\[
sol := q(t) = 50 - 50 e^{-\frac{t}{50}}
\]  \hspace{1cm} (13)

At the end of the 10th minute, the differential equation now changes to

\[
\text{new_de := diff(q(t),t) = 2*0 - 2*q(t)/100;}
\]
\[
\text{new_de := } \dot{q}(t) = -\frac{q(t)}{50}
\]  \hspace{1cm} (14)
while the initial value for the second equation is the mass of salt which was left when the first
differential equation quit.

\[ \text{new}\_\text{inits} := q(10) = 50 - 50 e^{-\frac{t}{5}} \]  \hspace{1cm} (15)

We can then dsolve the new differential equation to get the solution which holds after 10 minutes.

\[ q(t) = -e^{-\frac{t}{50}} \left( -50 + 50 e^{-\frac{1}{5}} \right) \frac{-\frac{t}{5}}{e^{-\frac{1}{5}}} \]

\[ \text{new}\_\text{sol} := q(t) = -50 \left( -1 + e^{-\frac{1}{5}} \right) e^{-\frac{t}{50}} + \frac{1}{5} \]  \hspace{1cm} (16)

We can then write the solution as a piecewise function

\[ \text{final}\_\text{sol} := \begin{cases} 
50 - 50 e^{-\frac{t}{50}} & 0 \leq t < 10 \\
-50 \left( -1 + e^{-\frac{1}{5}} \right) e^{-\frac{t}{50}} + \frac{1}{5} & 10 < t 
\end{cases} \]  \hspace{1cm} (17)

We can then graph the solution. As expected, since after 10 minutes the concentration of salt flowing in
goes to zero, then the concentration of salt in the tank will also decay exponentially to zero as the fresh
water flushed out the tank.

\[ \text{plot}(\text{final}\_\text{sol}, t=0..250); \]
We can take the Laplace transform of the solution by breaking the Laplace integral into two intervals.

\[
\int_0^{10} \left( 50 - 50 \exp \left( \frac{-t}{50} \right) \right) e^{-st} \, dt + \int_{10}^{\infty} -50 \left( -1 + e^{-\frac{1}{5}} \right) e^{-\frac{t}{50} + \frac{1}{5} s} e^{-st} \, dt
\]

(18)

To get the Laplace integral to converge, we need to take \( s > -\frac{1}{50} \). The value command computes the symbolic integrals.

\[
> \text{assume} (s > -\frac{1}{50}); \\
> \text{value(\%)}; \\
> \text{my_answer:=} Q(s) = \text{simplify(\%)};
\]

\[
\frac{50 \left( e^{10s} - 1 - 50 s + 50 e^{-\frac{1}{5} s} \right) e^{-10s}}{(1 + 50 s) s} - \frac{2500 \left( -1 + e^{-\frac{1}{5}} \right) e^{-10s}}{1 + 50 s}
\]
my_answer := Q(s) = \frac{50 \ e^{-10\ s} \ (e^{10\ s} - 1)}{(1 + 50\ s)\ s} \quad (19)

Of course, you can check your answer with the answer in BB:

\[
\begin{align*}
  & 1/(s\ (s+1/50)) - \exp(-10\ s)/(s\ (s+1/50)); \\
  & \text{book_answer} := Q(s) = \text{simplify(%)};
\end{align*}
\]

\[
\begin{align*}
  & \frac{1}{s \left(s + \frac{1}{50}\right)} - \frac{\ e^{-10\ s}}{s \left(s + \frac{1}{50}\right)} \\
  & \text{book_answer} := Q(s) = -\frac{50 \left(-1 + \ e^{-10\ s}\right)}{(1 + 50\ s)\ s} \quad (20)
\end{align*}
\]

The two answer agree.

\[
\begin{align*}
  & \text{rhs(my_answer)} - \text{rhs(book_answer)}; \\
  & \text{simplify(%)};
\end{align*}
\]

\[
\begin{align*}
  & \frac{50 \ e^{-10\ s} \ (e^{10\ s} - 1)}{(1 + 50\ s)\ s} + \frac{50 \left(-1 + e^{-10\ s}\right)}{(1 + 50\ s)\ s} \\
  & 0 \quad (21)
\end{align*}
\]

Here is an alternative solution using Laplace transform methods from the very start. You may note the advantage, since the solution is significantly shorter.

We can solve directly by moving the q(t) to the left hand side, then taking the Laplace transform of both sides and substituting the initial values.

\[
\begin{align*}
  & \text{restart:} \\
  & \text{with(inttrans):} \\
  & \text{alt_de} := \text{diff(q(t),t)+1/50*q(t)=piecewise(0<=t \ and \ t<10,1,t>10,0);} \\
  & \text{alt_inits} := q(0)=0; \\
  & \text{alt_de} := \frac{d}{dt} q(t) + \frac{1}{50} q(t) = \begin{cases} 
1 & 0 \leq t \ \text{and} \ t < 10 \\
0 & 10 < t
\end{cases} \\
  & \text{alt_inits} := q(0)=0 \quad (22)
\end{align*}
\]

We then take the Laplace transform. The answer agrees with what we got in the first solution.

\[
\begin{align*}
  & \text{alias(Q(s)=laplace(q(t),t,s))}; \\
  & \text{laplace(alt_de,t,s)}; \\
  & \text{subs(alt_inits,%)}; \\
  & \text{my_alt_sol} := Q(s) = \text{solve(%,Q(s))};
\end{align*}
\]
\[ sQ(s) - q(0) + \frac{1}{50} Q(s) = \frac{1 - e^{-10s}}{s} \]

\[ sQ(s) + \frac{1}{50} Q(s) = \frac{1 - e^{-10s}}{s} \]

\[ my_{\text{alt.sol}} := Q(s) = -\frac{50 \left(-1 + e^{-10s}\right)}{s \left(50s + 1\right)} \]