First we give the four step solution in Maple:

\[
\begin{align*}
\text{restart:} \\
f &:= \text{piecewise}(0 \leq t \text{ and } t < \frac{\pi}{2}, 1, t \geq \frac{\pi}{2}, 0) \\
f &:= \text{convert}(f, \text{Heaviside}) \\
\begin{cases} 
1 & 0 \leq t \text{ and } t < \frac{1}{2} \pi \\
0 & \frac{1}{2} \pi \leq t 
\end{cases} \\
f &:= \text{Heaviside}(t) - \text{Heaviside}\left(-\frac{1}{2} \pi + t\right)
\end{align*}
\]

(1)

\[
\begin{align*}
\text{with(inttrans):} \\
\text{alias}(Y(s) = \text{laplace}(y(t), t, s)): \\
d e &:= \text{diff}(y(t), t$2) + y(t) = f; \\
inits &:= y(0) = 0, D(y)(0) = 1; \\
\begin{align*}
d e &:= \frac{d^2}{dt^2} y(t) + y(t) = \text{Heaviside}(t) - \text{Heaviside}\left(-\frac{1}{2} \pi + t\right) \\
inits &:= y(0) = 0, D(y)(0) = 1
\end{align*}
\]

(2)

\[
\begin{align*}
\text{laplace}(de, t, s); \\
\text{subs}(\text{inits}, \%); \\
Y(s) &:= \text{solve}(\% , Y(s)); \\
\text{sol} &:= \text{invlaplace}(\%, s, t);
\end{align*}
\]

(3)

The Laplace transform of the right hand side of the equation can be computed by rewriting the piecewise function in terms of Heavisides as \((\text{new} - \text{old}) \text{Heaviside}(t - \text{transition})\).

The old is 0, and the new is 1, with the transition at \(t = 0\), so we get \((1 - 0) \text{Heaviside}(t - 0)\) for the first part. Then we note that the new is 0, the old is 1, and the transition is at \(t = \frac{\pi}{2}\). Hence, the second part is
This entire right hand side is

\[ \text{Heaviside}(t) - \text{Heaviside}\left(t - \frac{\pi}{2}\right) \]. Now since Heaviside(t) is always 1 to the right of t = 0, the

Laplace of Heaviside(t) is the same as the Laplace of 1, i.e., \( \frac{1}{s} \). One can use line 12 in Table 5.3.1 to

see that the Laplace transform of \( \text{Heaviside}\left(t - \frac{\pi}{2}\right) \) is \( \frac{e^{-\frac{\pi s}{2}}}{s} \). Hence, the Laplace transform of the

entire right hand side is \( \frac{1}{s} - \frac{e^{-\frac{\pi s}{2}}}{s} \cdot 1 \).

To find the inverse Laplace transform of \( -1 + \frac{1}{2} e^{-\frac{\pi s}{2}} - s \), we can break it into two parts,

\[ -\frac{1}{s} + \frac{1}{s^2+1} \] and \[ -\frac{e^{-\frac{\pi s}{2}}}{s(s^2 + 1)} \].

For the first part we can use partial fractions:

\[ \frac{s+1}{s(s^2+1)} \]

\[ \text{convert}(\%), \text{parfrac}, s) \]

\[ \frac{1}{s^2+1} + \frac{1}{s} \]

\[ \frac{1 - s}{s^2 + 1} + \frac{1}{s} - \frac{s}{s^2 + 1} + \frac{1}{s} \], from which we get \( \sin(t) - \cos(t) + 1 \), by lines 5, 6, and 1 on Table 5.3.1.

We hold the exponential in abeyance, and use parfrac on \( -\frac{1}{s(s^2+1)} \) to get \( \frac{s}{s^2+1} - \frac{1}{s} \), from which we get \( (\cos(t) - 1) \) Heaviside(t). Remembering that we also have an ignored exponential, \( e^{-\frac{\pi s}{2}} \), we then replace all the \( t \)’s by \( t - \frac{\pi}{2} \), giving \( \left[ \cos\left(t - \frac{\pi}{2}\right) - 1 \right] \) Heaviside\left(t - \frac{\pi}{2}\right). Using the standard trigonometric identity for the cosine of the complement of an angle, we can rewrite this as \( (\sin(t) - 1) \) Heaviside\left(t - \frac{\pi}{2}\right). Hence, putting it all together, we get

\[ \sin(t) - \cos(t) + 1 + (\sin(t) - 1) \text{Heaviside}\left(t - \frac{\pi}{2}\right) \].
\[ -\frac{1}{s(s^2+1)}; \]
\[
\text{convert(\%,parfrac)};\]
\[
-\frac{1}{s(s^2+1)} \]
\[
\frac{s}{s^2+1} - \frac{1}{s}\]

The system can be interpreted as a mass of 1 kg suspended from a Hooke's Law spring at the equilibrium position, but given an upwards velocity of 1 m/sec, and exposed to an upwards force of 1 Newton from time zero to time \(\pi/2 = 1.57\) sec. There is no damping, hence the lack of a \(y'\) term in the differential equation, so the oscillation will continue forever.

The external force is disconnected at \(t = 1.57\), so that the total response, in red, differs from the response in green that would have been seen if the external force on the rhs had never been connected.

\[ \text{new\_de:=lhs(de)=0}; \]
\[ \text{inits:=y(0)=0,D(y)(0)=1}; \]
\[ \text{laplace(new\_de,t,s)}; \]
\[ \text{subs(inits,%)}; \]
\[ \text{Y(s)=solve(%,Y(s))}; \]
\[ \text{alt\_sol:=invlaplace(%,s,t)}; \]

\[
\text{new\_de:=} \frac{d^2}{dt^2} y(t) + y(t) = 0\]
\[
\text{inits:=} y(0) = 0, D(y)(0) = 1\]
\[
\text{s}^2 \ Y(s) - D(y)(0) - s \ y(0) + Y(s) = 0\]
\[
\text{s}^2 \ Y(s) - 1 + Y(s) = 0\]
\[
Y(s) = \frac{1}{s^2 + 1}\]
\[ \text{alt\_sol:=y(t) = sin(t)}\]

The green solution is for the differential equation with the same inits but for temporary forcing term. As you can see, the additional upwards force raises the mass higher than it would rise against the spring if only the original velocity were taken into account. After 1.57 seconds, the force is turned off, but the mass continues to rise because of the velocity that it had acquired before the external force was turned off. Eventually, however, the spring manages to slow the mass and stop its upward motion, so it heads back down and oscillates.

\[ \text{plot([rhs(sol),f,sin(t)],t=0..12,color=[red,blue,green],}\]
\[ \text{scaling=constrained);} \]
Using the four step method in Maple, we get

```maple
> restart:
with(inttrans):
alias(Y(s)=laplace(y(t),t,s));
de:=diff(y(t),t$2)+4*y(t)=sin(t)-sin(t-2*Pi)*Heaviside(t-2*Pi);
inits:=y(0)=0,D(y)(0)=0;
laplace(de,t,s);
subs(inits,%);
Y(s)=solve(%,Y(s));
sol:=invlaplace(%,s,t);
```

\[
\begin{align*}
&Y(s) = \text{solve} \\
&de := \frac{d^2}{dt^2} y(t) + 4 y(t) = \sin(t) - \sin(t - 2\pi) \text{Heaviside}(t - 2\pi) \\
&\text{inits} := y(0) = 0, D(y)(0) = 0 \\
&\text{sol} := \text{invlaplace}(%, s, t);
\end{align*}
\]
To take the Laplace transform of the right hand side, we use lines 5 and 13 in Table 5.3.1 on \( \sin(t) \) and on \( -\sin(t) \) \( \text{Heaviside}(t - 2\pi) \), respectively.

In particular, for line 13, \( f \) is \( -\sin(t) \), so we get \( F = -\frac{1}{s^2 + 1} \), and the Laplace transform is \(-\frac{e^{-2s\pi}}{s^2 + 1}\).

To find the inverse Laplace transform of \( \frac{-1}{(s^2 + 1) (s^2 + 4)} \), we use partial fractions. Writing

\[
\frac{1}{3 (s^2 + 4)} - \frac{1}{3 (s^2 + 1)} \quad \text{as} \quad \frac{1}{6} - \frac{2}{3 (s^2 + 4)} - \frac{1}{3 (s^2 + 1)},
\]

we go backwards through line 5 of the table to get

\[
-\frac{1}{6}\sin(2t) + \frac{1}{3}\sin(t) = -\frac{1}{6}(\sin(2t) - 2\sin(t))
\]

as the inverse Laplace of \( \frac{-1}{(s^2 + 1) (s^2 + 4)} \).

For the part with the exponential, we hold the exponential in abeyance, then do partial fractions as before, getting

\[
\frac{1}{6}\sin(2t) - \frac{1}{3}\sin(t)
\]

Heaviside(t). Finally, we replace each \( t \) by \( (t - 2\pi) \), getting

\[
\left(\frac{1}{6}\sin(2(t - 2\pi)) - \frac{1}{3}\sin((t - 2\pi))\right)\text{Heaviside}(t - 2\pi).
\]

Since sine is \( 2\pi \) periodic, this term becomes

\[
\left(\frac{1}{6}\sin(2(t - 2\sin(t))\right)\text{Heaviside}(t - 2\pi).
\]

Putting the two pieces together, we get

\[
\frac{1}{6}(2\sin(t) - \sin(2t))(1 - \text{Heaviside}(t - 2\pi)).
\]

\[
> -1/((s^2+1)*(s^2+4));
\]
\[
\text{convert}(:, \text{parfrac}, \text{s});
\]
\[
- \frac{1}{(s^2 + 1) (s^2 + 4)}
\]
\[
- \frac{1}{3 (s^2 + 1)} + \frac{1}{3 (s^2 + 4)}
\]