MATH 409 Homework 2

1. For the sequence $4s_{n+1} = s_n^2 + 3$ determine its behaviour as $n \to \infty$ in each of the cases $s_1 < 1$, $1 < s_1 < 3$ and $s_1 > 3$. In each case prove that it has the claimed properties.

2. Let $t_1 > 0$ and define t_n by $t_{n+1} = \big(1 - 1/(2n^2)\big)t_n$ for $n \geq 1$. Show that t_n converges and find its limit.

3. Assume that $s_n \neq 0$ for all n and that $L = lim_{n\to\infty} |s_{n+1}/s_n|$ exists.
   i Show that if $L < 1$ then for any a such that $L < a < 1$, there is an N such that $|s_{n+1}| < a|s_n|$ for $n > N$. Hence show that $|s_n| < a^{n-N}|s_N|$ for $n \geq N$ and thus prove that s_n converges.
   ii Show that if $L > 1$ then $lim_{n\to\infty} |s_n| = \infty$.
   iii What can you conclude about the convergence of s_n if $L = 1$?

4. f is a function defined on the integers $\mathbb{Z}$, and $g(n) = f(n^2)$ for $n = 1, 2, 3, \ldots$.
   Show that if, (i) $f(n) \to L$ as $n \to \infty$, then (ii) $g(n) \to L$ as $n \to \infty$.
   Need the converse hold? That is, does (ii) imply (i)?

5. For each of the following, if true give a proof, if false provide a counterexample.
   i Let $\{s_n\}$ be a sequence such that $lim_{n\to\infty}(s_{n+1} - s_n) = 0$. Then s_n must converge.
   ii Let $\{s_n\}$ be a sequence such that $|s_{n+1} - s_n| < 1/n$ for all n. Then s_n must converge.
   iii If $a_n^2 \to A^2$ as $n \to \infty$ then either $a_n \to A$ as $n \to \infty$ or $a_n \to -A$ as $n \to \infty$.
   iv If $a_n^3 \to A^3$ as $n \to \infty$ then $a_n \to A$ as $n \to \infty$.
   v If $a_n^2 \to A^2$ and $a_{n+1} - a_n \to 0$ as $n \to \infty$, then $a_n \to A$ or $a_n \to -A$ as $n \to \infty$.