MATH 409 Homework 3

1. Suppose that $\alpha > 0$ and $a_0 > 0$. The sequence $\{a_n\}$ is defined by the relation

$$a_{n+1} = \frac{a_n^2 + \alpha}{2a_n}$$

Show that $\{a_n\}$ decreases for $n \geq 1$ and that $a_n \to \sqrt{\alpha}$ as $n \to \infty$.

2. We know from a previous problem that if $\{s_n\}$ is a sequence such that $|s_{n+1} - s_n| < 1/n$ for all n then it need not be the case that $\{s_n\}$ converges [$s_n = \mathbf{log}\, n$ is a counterexample].

   a Suppose now that $|s_{n+1} - s_n| < a^n$ for $a < 1$ and all for n. Show that

$$|s_{n+p} - s_n| < a^n(1 + a + a^2 + \ldots + a^{p-1}) < \frac{a^n}{1-a}$$

   Does $\{s_n\}$ converge?

   b [Extra credit]. If now $|s_{n+1} - s_n| < n^{-p}$, for what values of p will this condition ensure that $\{s_n\}$ converges?

3. Which of the following series converge (give reasons for your answers)

   a) $\sum \frac{\sin(n!)}{n^2}$ b) $\sum \frac{n^{10)}}{2^n}$ c) $\sum \sin(\frac{1}{n^2})$

4. For each of the following, if true give a proof, if false provide a counterexample. (Assume that $a_n \geq 0$ and $b_n \geq 0$ for all n.)

   i Let $\{d_n\}$ be a bounded sequence and $\sum c_n$ be convergent. Then $\sum c_n d_n$ converges.

   ii Let $\{d_n\}$ be a bounded sequence and $\sum c_n$ be absolutely convergent. Then $\sum c_n d_n$ converges.

   iii If $a_{n+1}/a_n < 1$ for all n then $\sum a_n$ converges.

   iv If $a_n - b_n \to 0$ as $n \to \infty$ and $\sum b_n$ converges, then $\sum a_n$ converges.

   v If $\sum a_n$ converges then $\sum a_n^2$ converges.

   vi If $\sum a_n^2$ converges then $\sum a_n$ converges.

   vii If $\sum a_n$ and $\sum b_n$ are convergent, then $\sum \sqrt{a_n b_n}$ is convergent.

   viii If $\sum a_n$ and $\sum b_n$ are divergent, then $\sum \sqrt{a_n b_n}$ is divergent.