MATH 409 Homework 6

1. For each of the following sequences of functions $\{f_n(x)\}$, determine whether they converge pointwise on $[0,1]$, and if so, find the limit function $f(x)$. Do they converge uniformly? For those that do not, determine whether $\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx$.

   a) $f_n(x) = (x - \frac{1}{n})^2$ b) $f_n(x) = n^\alpha(1-x)x^n$, $\alpha \geq 0$ c) $f_n(x) = \dfrac{nx}{1+n^2x^2}$.

2. For what values of α is the sequence of functions $s_n(x) = nxe^{-\frac{1}{2}n^\alpha x^2}$ uniformly convergent on $[0,1]$?

3. Test the following series for uniform convergence for $x \in [0,1]$.

   a) $\displaystyle\sum_1^\infty \frac{(1-x)x^n}{n}$ b) $\displaystyle\sum_1^\infty \frac{x}{n^{\frac{3}{4}} + n^{\frac{3}{2}}x^2}$ c) $\displaystyle(1-x)\sum_1^\infty \frac{x^n}{1+x^n}$.

4. For each of the following, if true give a proof, if false provide a counterexample.

   i If $f_n \to f$ and $g_n \to g$ uniformly on a set S then $f_n + g_n \to f + g$ uniformly on S.

   ii If $f_n \to f$ and $g_n \to g$ uniformly on a set S then $f_n g_n \to fg$ uniformly on S.

   iii If f_n is a sequence of bounded (but not necessarily continous) functions on $[a,b]$ that converges uniformly to f on $[a,b]$, then f is bounded on $[a,b]$.

   iv If f_n is a sequence of continous functions on $[a,b]$ that converges pointwise to f on $[a,b]$, then f is bounded on $[a,b]$.

   v If $\sum_1^\infty |f_n(x)|$ is uniformly convergent on $[a,b]$ then so also is $\sum_1^\infty f_n(x)$.

5. Let $f_n(x) = (-1)^n(1-x^2)x^n$ on $[0,1]$.

   i Show that for each $x \in [0,1]$, $\sum_1^\infty f_n(x)$ is absolutely convergent.

   ii Show that $\sum_1^\infty f_n(x)$ is uniformly convergent on $[0,1]$.

   iii Is $\sum_1^\infty |f_n(x)|$ uniformly convergent on $[0,1]$?