

MATH 409 Homework 7

1. Let $f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$ Is $f(x)$ differentiable at $x = 0$? Justify your answer.

2. Show that $\tan x > x$ for all $x \in (0, \frac{\pi}{2})$.

3. Let f be differentiable on \mathbb{R} with $1 \leq f'(x) \leq 2$ for all x and $f(0) = 0$. Prove that $x \leq f(x) \leq 2x$ for all x .

4. If the function $f(x)$ is differentiable at $x = c$ show that there exists a positive constant A and a $\delta > 0$ such that

$$|f(x) - f(c)| \leq A|x - c| \quad \text{if } |x - c| < \delta \quad (*)$$

If the function f satisfies $(*)$, does it imply that f is differentiable at $x = c$?

5. The function f satisfies $|f(x) - f(y)| \leq |x - y|^2$ for all real x and y . What can you say about f ?

6. If the functions f , g and h are continuous on $[a, b]$ and differentiable on $]a, b[$ show that for some $c \in]a, b[$

$$\begin{vmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(c) & g'(c) & h'(c) \end{vmatrix} = 0$$

7. Let f be differentiable on \mathbb{R} with $A = \sup|f'(x)|; x \in \mathbb{R}$ and $A < 1$. For a given value of a_0 , define the sequence a_n by $a_n = f(a_{n-1})$, $n \geq 1$.

a. Show that $|a_{n+1} - a_n| \leq A|a_n - a_{n-1}|$ for $n \geq 1$.

b. Deduce that $\{a_n\}$ is a cauchy sequence and hence convergent.

c. Deduce that the graph of $f(x)$ must cross the line $y = x$.

d. How many times can it do so?