MATH 409 Homework 8

1. Suppose that f is continuous on $[a, b]$ and $\int_a^b fg = 0$ for all continuous functions g on $[a, b]$. Prove that $f = 0$.

   Suppose now that g satisfies the additional constraint $g(a) = g(b) = 0$. Does it again follow that f must be identically zero?

2. True or false: If $f(x) \geq 0$ for all $x \in [a, b]$ and $f(x) > 0$ for some $x \in [a, b]$, then $\int_a^b f > 0$.

3. Let f be a bounded function on $[a, b]$ and let P be a partition of $[a, b]$. Let $M_i = \max_{x\in[t_{i-1},t_i]}\{f(x)\}$ and $m_i = \min_{x\in[t_{i-1},t_i]}\{f(x)\}$ with M_i' and m_i' the corresponding values for $|f|$.

   a. Prove that $M_i' = \max\{|M_i|, |m_i|\}, \; m_i' = \min\{|M_i|, |m_i|\}$.

   b. Prove that $M_i' - m_i' \leq M_i - m_i$.

   c. Prove that if f is integrable on $[a, b]$ then so is $|f|$.

   d. Conclude also that $|\int_a^b f| \leq \int_a^b |f|$.

   e. Does the converse hold, that is, if $|f|$ is integrable on $[a, b]$ then so is f?

4. Let f be a bounded function on $[a, b]$, with say $|f(x)| \leq B$, and P a partition of $[a, b]$.

   a. Prove that $U(f^2, P) - L(f^2, P) \leq 2B(U(f, P) - L(f, P))$.

   b. Hence show that if f is integrable on $[a, b]$ then so also is f^2.

5. If x, y are real, show that $4xy = (x+y)^2 - (x-y)^2$ and $\max\{x, y\} = \frac{1}{2}(x+y-|x+y|)$. Prove that if f and g are integrable on $[a, b]$ then so also is fg and $\max\{f, g\}$.