MATH 409 Homework 9

[1] Test the following integrals for convergence (assume $\alpha > 0$)

(a) $\displaystyle\int_1^2 \frac{dx}{x(\log x)^\alpha}$ (b) $\displaystyle\int_2^\infty \frac{\sin x}{(\log x)^\alpha}\,dx$ (c) $\displaystyle\int_0^\infty \cos x^\alpha\,dx$ (d) $\displaystyle\int_0^\infty \frac{\sin x(1-\cos x)}{x^\alpha}\,dx$

[2] Discuss the convergence of the improper integrals

$$\int_0^\infty \frac{x^\alpha e^{-\beta x}\log x}{1+x^2}\,dx \qquad\qquad \int_0^\infty \frac{x^\alpha \sin x}{1+x^\beta}\,dx$$

for all real α, β.

[3] Prove that $\displaystyle\int_0^1 \frac{e^{-xy}}{1+y}\,dy$ has a limit as both $x \to 0^+$ and $x \to \infty$ and find these values.

[4] Consider the function $f(x) = x\displaystyle\sum_1^\infty \frac{1}{n^2+x^2}$. If the partial sums of the series defining f are written as $f_N(x)$, show that for each N and $x > 0$

$$f_N(x) - f_1(x) \le \int_1^N \frac{x}{t^2+x^2}\,dt \le f_N(x) \le \int_0^N \frac{x}{t^2+x^2}\,dt$$

Deduce that $\lim_{x\to\infty} f(x) = \frac{\pi}{2}$

[5] Use the Taylor expansion of $\log(1-t)$ to show that

$$\int_0^1 \frac{\log x}{1-x}\,dx = -\sum_1^\infty \frac{1}{n^2}$$

justifying your calculations.