**Theorem.** *(Taylor's Theorem). Let $f$ be $n$-times differentialble on $[a, b]$. Then*

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2!}f''(a)(b-a)^2 + \dots \frac{1}{(n-1)!}f^{(n-1)}(a)(b-a)^{n-1} + R_n$$
$$= \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k + R_n$$

*where $R_n$ is*

$$R_n = \begin{cases} \frac{f^{(n}(\eta_1)}{n!}(b-a)^n & \text{(Lagrange)} \\ \frac{f^{(n}(\eta_2)}{(n-1)!}(b-a)(b-\eta_2)^{n-1} & \text{(Cauchy)} \\ \int_a^b \frac{f^{(n)}(t)}{n!}(b-t)^n \, dt & \text{(Integral)} \end{cases}$$

*with $a < \eta_1, \eta_2 < b$.*

**Proof**. For a positive integer $p$, define $F(x)$ and $G(x)$ on $[a, b]$ by

$$F(x) = f(b) - f(x) - (b-x)f'(x) - \frac{(b-x)^2}{2!}f''(x) - \dots - \frac{(b-x)^{n-1}}{(n-1)!}f^{(n-1)}(x),$$
$$G(x) = (b-x)^p.$$

From the assumptions, $F$ is differentiable on $[a, b]$ and so

$$F'(x) = -f'(x) - (b-x)f''(x) - \dots - \frac{(b-x)^{n-2}}{(n-2)!}f^{(n-1)}(x) - \frac{(b-x)^{n-1}}{(n-1)!}f^{(n)}(x)$$
$$+ f'(x) + (b-x)f''(x) + \dots + \frac{(b-x)^{n-2}}{(n-2)!}f^{(n-1)}(x)$$
$$= -\frac{(b-x)^{n-1}}{(n-1)!}f^{(n)}(x)$$

(Here the first line is obtained by differentiating the derivatives of $f$ while the second by differentiating the powers of $(b-x)$).
Clearly, $G$ is differentiable on $[a, b]$ and

$$G'(x) = -p(b-x)^{p-1}$$

so that for $x \in (a, b)$, $G'(x) \neq 0$. Also, $F(b) = G(b) = 0$. Thus by the Cauchy mean value theorem there is a point $\eta \in (a, b)$ such that

$$\frac{F(a)}{G(a)} = \frac{F(b) - F(a)}{G(b) - G(a)} = \frac{F'(\eta)}{G'(\eta)} = \frac{(b-\eta)^{n-p}}{p(n-1)!}f^{(n)}(\eta).$$

Hence

$$F(a) = \frac{(b-\eta)^{n-p}(b-a)^p}{p(n-1)!} f^{(n)}(\eta) = R_n$$

By taking $p = n$ and $p = 1$ in the above we obtain respectively Lagrange's and Cauchy's form of the remainder. Note, that since the point $\eta$ will depend on $G$ and hence on $p$, we should expect different values of $\eta$ in the remainders.

Finally, we have by the Fundamental Theorem of Calculus that $F(b) - F(a) = \int_a^b F'(x)\,dx$ and inserting our above value for $R_n$ and calculation for $F'$ in here (remembering that $F(b) = 0$) we obtain

$$-R_n = -F(a) = F(b) - F(a) = \int_a^b F'(x)\,dx = -\int_a^b \frac{(b-x)^{n-1}}{(n-1)!} f^{(n)}(x)\,dx$$

This immediately gives the integral form of the remainder.

Note that the key step in the proof (other than the choice of $F$ and $G$ was the Mean Value Theorem - Taylor's Theorem is just the mean value theorem extended to higher derivatives..