Example Sheet 2

• **Formal definition of limit:** We say that
  \[ \lim_{x \to a} f(x) = L \]
  if for every \( \epsilon > 0 \), there exists a positive number \( \delta \) (depending on \( \epsilon \) and \( a \)) such that
  \[ |f(x) - L| < \epsilon \]
  whenever
  \[ 0 < |x - a| < \delta. \]

• **Right limit:** For every \( \epsilon > 0 \), there exists a positive number \( \delta \) (depending on \( \epsilon \) and \( a \)) such that
  \[ |f(x) - L| < \epsilon \]
  whenever
  \[ a < x < a + \delta. \]
  Notation: \( \lim_{x \to a^+} f(x) = L \)

• **Left limit:** For every \( \epsilon > 0 \), there exists a positive number \( \delta \) (depending on \( \epsilon \) and \( a \)) such that
  \[ |f(x) - L| < \epsilon \]
  whenever
  \[ a - \delta < x < a. \]
  Notation: \( \lim_{x \to a^-} f(x) = L \)

**Important Definitions:** Continuity of a function at a point [see pages 112 and 113 of the text], and continuity of a function on an interval [page 114].

**Two Important Theorems:** The Squeeze Theorem (also known as the Sandwich Principle) [page 98] and the Intermediate Value Theorem [page 118].

**Examples**

1. Sketch the graph of the function

   \[ f(x) = \begin{cases} 
   2 - x, & \text{if } x < -1; \\
   x, & \text{if } -1 \leq x < 1; \\
   4, & \text{if } x = 1; \\
   4 - x, & \text{if } x > 1. 
   \end{cases} \]

   Use the graph to state the value of each of the following limits, if it exists.
   (i) \( \lim_{x \to -1^-} f(x) \)  (ii) \( \lim_{x \to -1^+} f(x) \)  (iii) \( \lim_{x \to -1} f(x) \)  (iv) \( \lim_{x \to 1^-} f(x) \)  (v) \( \lim_{x \to 1^+} f(x) \)  (vi) \( \lim_{x \to 1} f(x) \)
2. The greatest integer function is defined by \([x] = \text{the largest integer that does not exceed } x\).
   The domain of this function is the set of all real numbers.
   (i) Sketch the graph of the function.
   (ii) Find all real numbers \(a\) for which \(\lim_{x \to a} [x]\) does not exist.
   (iii) Find all real numbers \(a\) where the function is discontinuous.
   (iv) Consider each of the points where the function is discontinuous. Is the function right continuous at such a point? left continuous? neither?

3. Let \(F(x) = \frac{x^2 - 1}{|x - 1|}\).
   (i) Sketch the graph of \(F\).
   (ii) Find \(\lim_{x \to 1^-} F(x)\).
   (iii) Find \(\lim_{x \to 1^+} F(x)\).
   (iv) Does \(\lim_{x \to 1} F(x)\) exist? Explain.

4. Study Example 13 on page 98.

5. Study Example 4 on page 83.

6. Show that \(\lim_{x \to 0} \cos(1/x)\) does not exist.

7. Find numbers \(a\) and \(b\) such that
   \[
   \lim_{x \to 0} \frac{\sqrt{ax + b} - 2}{x} = 1.
   \]

8. Find numbers \(a\) and \(b\) such that
   \[
   \lim_{x \to a} \frac{x^2 + ax - b}{x - a} = 2.
   \]

The next pair of examples is taken from the book Basic Analysis: Japanese Grade 11, translated and published by the American Mathematical Society, as part of the University of Chicago School Mathematics Project.

9. Find the values of the constants \(a\) and \(b\) such that
   \[
   \lim_{x \to 2} \frac{x^2 - ax + 8}{x^2 - (b + 2)x + 2b} = \frac{1}{5}.
   \]

10. Find the cubic function \(f(x)\) (to wit, a function of the form \(f(x) = ax^3 + bx^2 + cx + d\), where \(a, b, c,\) and \(d\) are constants) which satisfies the following conditions:
    \[
    \lim_{x \to 0} \frac{f(x)}{x} = 2 \quad \text{and} \quad \lim_{x \to 1} \frac{f(x)}{x - 1} = 1.
    \]
11. Let
\[ f(x) := \begin{cases} 
2x, & \text{if } x < 1; \\
 cx^2 + d, & \text{if } 1 \leq x \leq 2; \\
4x, & \text{if } x > 2. 
\end{cases} \]
Find the values of the numbers \( c \) and \( d \) that make \( f \) continuous at every point on the real line.

12. Define
\[ g(x) := \begin{cases} 
2 - x, & \text{if } 0 \leq x \leq 1; \\
- (x - 1)^2, & \text{if } 1 < x \leq 2. 
\end{cases} \]
Note that \( g(0) = 2 \) and \( g(2) = -1 \), so that 0 lies between \( g(0) \) and \( g(2) \). Is there a number \( c \) in the interval \((0, 2)\) such that \( g(c) = 0 \)? If so, explain why. If not, explain why this does not contradict the Intermediate Value Theorem.

13. Study the definitions of infinite limits (the informal as well as formal versions) on pages 86, 87, 108, and 109. Infinite one-sided limits may be defined in a similar fashion. Consider the following simple examples. Even though one encounters an infinite limit in each case, it is important to note the essential qualitative difference between the behaviour of the two functions near the origin.

(i) Find \( \lim_{x \to 0^+} \frac{1}{x} \).
(ii) Find \( \lim_{x \to 0^-} \frac{1}{x} \).
(iii) Find \( \lim_{x \to 0^+} \frac{1}{x^2} \).
(iv) Find \( \lim_{x \to 0^-} \frac{1}{x^2} \).
(v) What is \( \lim_{x \to 0^+} \frac{1}{x} \)?
(vi) What is \( \lim_{x \to 0^-} \frac{1}{x^2} \)?