Exercise Set 3

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1. Let \( F(x) = \frac{x^2 - 1}{|x - 1|} \).
   (i) Sketch the graph of \( F \).
   (ii) Find \( \lim_{x \to 1^-} F(x) \).
   (iii) Find \( \lim_{x \to 1^+} F(x) \).
   (iv) Does \( \lim_{x \to 1} F(x) \) exist? Explain.

2. Find numbers \( a \) and \( b \) such that
   \[
   \lim_{x \to a} \frac{x^2 + ax - b}{x - a} = 2.
   \]

The next pair of examples is taken from the book Basic Analysis: Japanese Grade 11, translated and published by the American Mathematical Society, as part of the University of Chicago School Mathematics Project.

3. Find the values of the constants \( a \) and \( b \) such that
   \[
   \lim_{x \to 2} \frac{x^2 - ax + 8}{x^2 - (b + 2)x + 2b} = \frac{1}{5}.
   \]

4. Find the cubic function \( f(x) \) (to wit, a function of the form \( f(x) = ax^3 + bx^2 + cx + d \), where \( a, b, c, \) and \( d \) are constants) which satisfies the following conditions:
   \[
   \lim_{x \to 0} \frac{f(x)}{x} = 2 \quad \text{and} \quad \lim_{x \to 1} \frac{f(x)}{x - 1} = 1.
   \]

5. (i) Give a proof of the Sandwich Principle/Squeeze Theorem stated and discussed in lecture.
   (ii) Prove the following version of the Squeeze Theorem: Suppose that \( f(x) \leq g(x) \leq h(x) \) for every \( x > A \) (where \( A \) is some fixed real number). If \( \lim_{x \to \infty} f(x) = \lim_{x \to \infty} h(x) = L \) (where \( L \) is a real number), then \( \lim_{x \to \infty} g(x) = L \) as well.
   (iii) Use the theorem from part (ii) to evaluate \( \lim_{x \to \infty} \frac{\sin^2 x}{x^2} \).