§1.2. Concerning subsets of the complex plane

In this section we define some special types of subsets of the complex plane; they will play a basic role in subsequent analysis.

Definition 1.2.1. Suppose that R is a fixed positive number, and that z_0 is a fixed complex number.
(i) The circle of radius R centred at z_0 is defined by

$$C(z_0; R) := \{z \in \mathbf{C} : |z - z_0| = R\}\,.$$

(ii) The *open disc* of radius R centred at z_0 is defined by

$$D(z_0; R) := \{z \in \mathbf{C} : |z - z_0| < R\}\,.$$

(iii) The *closed disc* of radius R centred at z_0 is defined by

$$\overline{D}(z_0; R) := \{z \in \mathbf{C} : |z - z_0| \leq R\} = D(z_0; R) \cup C(z_0; R)\,.$$

Definition 1.2.2. (i) Let A be a subset of the complex plane $\mathbf{C}$. The *complement of* A is the set of all complex numbers that are not in A, to wit, $A^c := \mathbf{C} \setminus A := \{z \in \mathbf{C} : z \notin A\}$.
(ii) Suppose that A is a nonempty subset of $\mathbf{C}$. We say that A is *open* if for every $w \in A$, there is some positive number R_w (the subscript denoting the possible dependence of R_w on w) such that $D(w; R_w) \subseteq A$.
(iii) A subset A of $\mathbf{C}$ is said to be *closed* if A^c is open.
(iv) A subset A of $\mathbf{C}$ is said to be *bounded* if there is some positive number Δ such that $A \subseteq \overline{D}(0; \Delta)$, *i.e.*, $|z| \leq \Delta$ for every $z \in A$. A set is said to be *unbounded* if it is not bounded.
(v) A subset of the complex plane is said to be *compact* if it is closed and bounded.

Remark 1.2.3. (i) That the entire complex plane $\mathbf{C}$ is an open set follows directly from definition. The empty set is also open (vacuously). Consequently, both $\mathbf{C}$ and the empty set are also closed (because they are complements of each other). Although we will not prove this fact, we remark that these are the only two subsets of $\mathbf{C}$ that have this property (namely being open and closed).
(ii) Suppose that A is a nonempty subset of $\mathbf{C}$. The reader will confirm that A is open if and only if for every $w \in A$, there is a positive number r_w such that $\overline{D}(w; r_w) \subseteq A$.
(iii) Suppose that A is as above. Then A is *not* open if and only if there is some $w_0 \in A$ such that $D(w_0; R) \nsubseteq A$ for *every* positive number R, that is, $D(w_0; R) \cap A^c \neq \emptyset$ for every $R > 0$.
(iv) Every disc in the complex plane (open or closed) is a bounded set. A closed disc is a compact set.
(v) A subset A of $\mathbf{C}$ is unbounded if and only if for every positive number T, there is an element $z_T \in A$ such that $|z_T| > T$. Each of the following sets is unbounded: (a) the real axis, (b) $\mathbf{C}$, (c) $\{1 + ni : n \text{ a positive integer}\}$.

Example 1.2.4. (i) Suppose that R is a fixed positive number, and that $z_0 \in \mathbf{C}$. We show that $D(z_0; R)$ is an open set (hence the term 'open' disc). Let $w \in D(z_0; R)$, so that $R_w := R - |w - z_0| > 0$ by definition. If $z \in D(w; R_w)$, then the triangle inequality gives the estimate

$$|z - z_0| = |z - w + w - z_0| \leq |z - w| + |w - z_0| < R_w + |w - z_0| = R\,.$$

This shows that $D(w; R_w) \subseteq D(z_0; R)$. As the choice of w was arbitrary, we conclude that $D(z_0; R)$ is an open set.
(ii) The *upper-half plane*

$$\mathbf{H}_{\uparrow} := \{z \in \mathbf{C} : \Im(z) > 0\}$$

is an open set. Let $w = \sigma + it \in \mathbf{H}_{\uparrow}$, so that $t > 0$. If $z = x + iy \in D(w; t)$, then $|z - w| < t$, so Proposition 1.1.5 implies the inequality $|y - t| < t$. Therefore $y = t - (t - y) > 0$, *i.e.*, $z \in \mathbf{H}_{\uparrow}$, and it follows that $D(w; t) \subseteq \mathbf{H}_{\uparrow}$. This being true for every $w \in \mathbf{H}_{\uparrow}$, we conclude that $\mathbf{H}_{\uparrow}$ is an open set.
(iii) The set $\overline{D}(z_0; R)$ is not open (for any choice of $z_0 \in \mathbf{C}$ and $R > 0$). Indeed, if w_0 is any number with $|w_0 - z_0| = R$, then $D(w_0; r)$ intersects the complement of $\overline{D}(z_0; R)$ for every choice of $r > 0$. The same argument shows that the circle $C(z_0; R)$ is not an open set.
(iv) The reader is asked to prove that the set $U := \{z = x + iy : x > 0,\ y > 0\}$ is an unbounded, open subset of the complex plane.
(v) The reader will verify that the following sets are closed: $A := \{z \in \mathbf{C} : |z| \geq R\}$ and $B := \{z = x + iy : x \leq 0\}$.

Definition 1.2.5. (i) Suppose that $A \subseteq \mathbf{C}$ is a nonempty set. We say that A is *disconnected* if there exist open sets U and V such that $A \subseteq U \cup V$, both $A \cap U$ and $A \cap V$ are nonempty sets, and $A \cap U \cap V = \emptyset$. The sets U and V are said to provide a *disconnection* for A.
(ii) A subset A of the complex plane is said to be *connected* if it is not disconnected.

Example 1.2.6. The set $A := \{z = x + iy : xy = 1\}$ is a disconnected subset of the complex plane. The reader will check that the sets $U := \{z = x + iy : x > 0,\ y > 0\}$ and $V := \{z = x + iy : x < 0,\ y < 0\}$ are open, and that they provide a disconnection for A.

Definition 1.2.7. (i) Given complex numbers z and w, we denote by $[z, w]$ the (directed) line segment which starts at z and ends at w. A *polygonal curve* is a finite union of line segments $[z_0, z_1] \cup [z_1, z_2] \cup \cdots \cup [z_{n-1}, z_n]$. Note that each segment (save the first) begins where the preceding one ends.
(ii) Suppose that S is a nonempty subset of $\mathbf{C}$. We say that S is *polygonally connected* if every pair of points in S can be joined by a polygonal curve which is contained entirely in S.

The relationship between connectedness and polygonal connectedness is brought out in the next pair of results, the first of which is stated without proof.

Theorem 1.2.8. *Suppose that S is a nonempty subset of the complex plane. If S is polygonally connected, then S is connected.*

Even though a connected set need not be polygonally connected in general, the the two notions coincide for open sets:

Theorem 1.2.9. *If a (nonempty) set S is connected and open, then it is polygonally connected.*

Proof. Let $z_0 \in S$. Define

$$U := \{z \in S : z \text{ can be joined to } z_0 \text{ by a polygonal curve lying entirely in } S\}$$

and

$$V := \{z \in S : z \text{ cannot be joined to } z_0 \text{ by a polygonal curve lying entirely in } S\}.$$

If $w \in U \subseteq S$, then the openness of S provides a positive number R_w such that $D(w; R_w) \subseteq S$. Now if $z \in D(w; R_w)$, then the line segment starting at z and ending at w lies within $D(w; R_w) \subseteq S$. As w belongs to U, it can be joined to z_0 by a polygonal curve contained entirely within S. Thus z can be joined to z_0 by a polygonal curve contained entirely within S. This shows that $D(w; R_w) \subseteq U$, and hence that U is open.

We assert that V is also an open set. To that end let $w \in V \subseteq S$. As before there is a positive number r_w such that $D(w; r_w) \subseteq S$ (because S is open). If this disc is not contained in V, then it must contain a point, say z, which is not in V. As $D(w; r_w)$ is contained in S, z must belong to $S \setminus V$. Therefore z is a point in S which can be joined to z_0 by a polygonal curve contained in S. Once again, as the line segment joining w to z is contained in $D(w; r_w) \subseteq S$, it follows that w can be joined to z_0 by a polygonal curve contained in S, but this contradicts the assumption that $w \in V$. Therefore $D(w; r_w) \subseteq V$, hence V is open.

Now U and V are subsets of S by definition, so S contains their union as well. On the other hand, if z is any point in S, then it can either be joined to z_0 by a polygonal curve contained in S (in which case $z \in U$), or it cannot (in which case $z \in V$). So S is contained in the union of U and V, and we conclude that $S = U \cup V$. As $U \cap V$ is empty, $S \cap U \cap V = \emptyset$ as well. As $z_0 \in S$ and S is open, there is a positive number δ such that $D(z_0; \delta) \subseteq S$. Every point in this disc can be joined to the centre z_0 by a line segment (in particular a polygonal curve) which lies within the disc (hence within S). Therefore $S \cap U$ is nonempty. Thus we have a pair of open sets U and V such that $S = U \cup V$, $S \cap U \neq \emptyset$, and $S \cap U \cap V = \emptyset$. So the connectedness of S ensures that $S \cap V$ must be empty (otherwise U and V would provide a disconnection for S). As $V \subseteq S$, this means that V must be empty, that is, there is no point in S which cannot be joined to z_0 by a polygonal curve contained in S. In other words, every point in S can be joined to z_0 by a polygonal curve contained entirely in S. Finally, if z_1 and z_2 are any two points in S, then each of them can be joined to z_0 by a polygonal curve contained in S, hence z_1 and z_2 can themselves be joined by a polygonal curve contained entirely in S. Thus S is polygonally connected. ∎

Theorems 1.2.8 and 1.2.9 can be combined to give the following.

Corollary 1.2.10. *A nonempty open subset of the complex plane is connected if and only if it is polygonally connected.*

As sets of the type described above play an important role in complex analysis, we shall find it convenient to give them a name.

Definition 1.2.11. A nonempty open, connected (hence polygonally connected) subset of the complex plane is called a *region*.

Remark 1.2.12. The notions of connectedness and openness are not related to one another; in particular, it is possible for a set to be open and disconnected or to be connected and not open. Hence there is no redundancy in the preceding definition.

Example 1.2.13. (i) The right-half plane

$$\mathbf{H}_{\rightarrow} := \{z = x + iy \,:\, x > 0\}$$

is a region.
(ii) The set $A := \{z = x + iy \,:\, y \neq 0\}$ is an open set which is not connected. Hence it is not a region.

(iii) The closed disc $\overline{D}(0;1)$ is polygonally connected, hence connected. However it is not a region because it is not open.
(iv) The open disc $D(z_0;R)$ is a region for any choice of $z_0 \in \mathbf{C}$ and $R > 0$.
(v) Let a and b be fixed positive numbers. The set

$$S := \{z = x + iy \,:\, |x| < a,\ |y| < b\}$$

is a region. This is the interior of a rectangle bounded by $x = \pm a$ and $y = \pm b$.