1. Suppose $f$ is analytic in the disc $D(a; R)$, and define

$$g(z) := \begin{cases} \frac{f(z)-f(a)}{z-a}, & \text{if } 0 < |z-a| < R; \\ f'(a), & \text{if } z = a. \end{cases}$$

Show that $g$ is analytic throughout $D(a; R)$.

The next example outlines a proof of the Carathedory-Schwarz Lemma:

2. Suppose that $f$ is analytic in $D(0; 1)$ and that it obeys the conditions $f(0) = 0$ and $|f(z)| \leq 1$ for every $z \in D(0; 1)$.

(i) Define

$$g(z) := \begin{cases} \frac{f(z)}{z}, & \text{if } 0 < |z| < 1; \\ f'(0), & \text{if } z = 0. \end{cases}$$

Use Example 1 to conclude that $g$ is analytic in $D(0; 1)$.

(ii) Fix $z \in D(0; 1)$ and consider any $r$ satisfying $|z| < r < 1$. Apply the (corollary of) the Maximum-Modulus Theorem to the closed disc $\overline{D}(0; r)$ to deduce that $|g(z)| \leq \frac{1}{r}$. Allow $r$ to approach 1 and conclude that $|g(z)| \leq 1$.

(iii) Use (ii) to show that $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for every $z \in D(0; 1)$.

(iv) Prove: If $|f'(0)| = 1$ or if there is some $z_0 \in D(0; 1) \setminus \{0\}$ such that $|f(z_0)| = |z_0|$, then $f$ must be a function of the type $f(z) = e^{i\theta}z$, $z \in D(0; 1)$, where $\theta$ is some fixed real number.

3. (i) Use the Taylor expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1,$$

to evaluate the following sums:

$$\sum_{n=1}^{\infty} n z^n \quad \text{and} \quad \sum_{n=1}^{\infty} n^2 z^n, \quad |z| < 1.$$

(ii) Put $z = i/2$ in the last sum above and compute the sum of the following series:

$$\sum_{k=1}^{\infty} \frac{(-1)^k k^2}{2^{2(k-1)}}.$$

4. (i) Obtain Laurent series expansions in powers of $z$ for the function

$$f(z) = \frac{1}{z^2(1-z)}$$
in each of the following domains:

\[ D_1 := \{ z \in \mathbb{C} : 0 < |z| < 1 \} \quad \text{and} \quad D_2 := \{ z \in \mathbb{C} : 1 < |z| < \infty \}. \]

(ii) Use (i) to evaluate the integral

\[ \int_{C(0;1/2)} f(z) \, dz. \]

5. (i) Find the Laurent series expansions in powers of \( z - 1 \) for the function

\[ f(z) = \frac{z}{(z - 1)(z - 2)} \]

in each of the following domains:

\[ D_1 := \{ z \in \mathbb{C} : 0 < |z - 1| < 1 \} \quad \text{and} \quad D_2 := \{ z \in \mathbb{C} : 1 < |z - 1| < \infty \}. \]

(ii) Suppose \( C \) is a positively oriented, simple closed contour contained in the region \( D_1 \). Compute the following integrals:

\[ \int_C \frac{f(z)}{(z - 1)^k} \, dz, \]

where \( k \) is an arbitrary integer.

6. Use Laurent series expansions to compute the residues of the function

\[ f(z) = \frac{2z + 1}{(z + 1)^2(z + 2)} \]

at each of its singularities. Confirm your findings via other means.

7. Suppose that \( f \) has a pole of order \( m \) at a point \( z_0 \). Shew that

\[ \lim_{z \to z_0} (z - z_0)^k f(z) = \infty, \]

for every nonnegative integer \( k \) less than \( m \). What can be said if \( k \geq m \)?

8. Suppose that \( p \) is a fixed integer. Show that the function

\[ g(z) = \frac{e^{-z} \sin^2(\pi z)}{(z - p)^3} \]

has a pole at \( z = p \). Determine the order of the pole and compute \( \text{Res}\{g;p\} \).

9. Evaluate

\[ \int_C \frac{1}{(3z^2 + i)(2z - 1)^3} \, dz, \]

where \( C \) is the positively oriented square with vertices at \( z = \pm 1 \) and \( z = \pm i \).
10. Evaluate the following integrals:

(i) \[ \int_0^\pi \frac{dt}{2 - \cos^2 t} \]

(ii) \[ \int_0^{2\pi} \frac{d\theta}{a + b \sin \theta} \]

where \( a \) and \( b \) are real numbers with \( a > |b| \).

(iii) \[ \int_{-\infty}^\infty \frac{x \sin(\pi x)}{x^2 + 2x + 5} \, dx := \lim_{R \to \infty} \int_{-R}^R \frac{x \sin(\pi x)}{x^2 + 2x + 5} \, dx \]

Consider the function \( z \mapsto \frac{z \exp(iz \pi)}{z^2 + 2z + 5} \) along with the contour comprising the following pieces: the line segment (along the real axis) from \( z = -R \) to \( z = R \), followed by the circular arc from \( z = R \) to \( z = -R \) (traversed counterclockwise).