Example Sheet 11b

1. The purpose of this example is to compute the following integral:

\[ \int_0^\infty \frac{t^p}{1+t^2} \, dt, \]

where \(-1 < p < 1\) is fixed.

Let \( D := \mathbb{C} \setminus \{ iy : y \leq 0 \} \) and let \( z \mapsto \log(z) \) denote the unique branch of the logarithm defined on \( D \), satisfying the condition \( \log(1) = 0 \). Let \( z \mapsto z^p \) denote the corresponding branch of the power function on \( D \), that is, \( z^p := \exp(p \log(z)), z \in D \). Define

\[ f(z) := \frac{z^p}{1+z^2}. \]

Let \( \epsilon \) and \( R \) be positive numbers with \( \epsilon < R \). Consider the contour \( C \) comprising the following four pieces: the straight-line segment \( L_1 \) (along the real axis) from \( z = \epsilon \) to \( z = R \), followed by the semi-circular arc \( \Gamma_R \) (traversed in the counterclockwise direction) from \( z = R \) to \( z = -R \), followed by the line segment \( L_2 \) (along the real axis) from \( z = -R \) to \( z = -\epsilon \), followed by the semi-circular arc \( \gamma_\epsilon \) (traversed in the clockwise direction) from \( z = -\epsilon \) to \( z = \epsilon \).

(i) Compute

\[ \int_C f(z) \, dz. \]

(ii) Parametrize the straight-line segments in \( C \) to obtain the following:

\[ [1 + (-1)^p] \int_\epsilon^R \frac{t^p}{1+t^2} \, dt = \pi \exp\left(\frac{ip\pi}{2}\right) - \int_{\gamma_\epsilon} f(z) \, dz - \int_{\Gamma_R} f(z) \, dz. \]

(iii) Show that the integrals of \( f \) along \( \gamma_\epsilon \) and \( \Gamma_R \) tend to zero as \( \epsilon \to 0^+ \) and \( R \to \infty \), respectively.

(iv) Show that

\[ \int_0^\infty \frac{t^p}{1+t^2} \, dt = \lim_{R \to \infty} \int_\epsilon^R \frac{t^p}{1+t^2} \, dt = \frac{\pi}{2 \cos \left(\frac{p\pi}{2}\right)}. \]

2. Suppose that \( 0 < \alpha < 1 \) is fixed. Use Example 1 to deduce that

\[ \int_0^\infty \frac{x^{\alpha-1}}{1+x} \, dx = \frac{\pi}{\sin(\pi\alpha)}. \]