Example Sheet 4c

1. Let $\log(z)$ denote a branch of the logarithm defined on the region $D := \mathbb{C} \setminus \{x(1+i) : x \geq 0\}$. Assume that $\log(i) = 5\pi i/2$.
   (i) Evaluate $\log(-1)$, $\log(-2i)$, $\log(1-i)$, and $\log(3)$.
   (ii) Find the image of the lower-half plane $H := \{z \in \mathbb{C} : \Im(z) < 0\}$ under the mapping $z \mapsto \log(z)$.

2. (i) Find all possible values of $(-1)^{1/\pi}$.
   (ii) Find the principal value of $\left(\frac{e}{2}(-1 - \sqrt{3}i)\right)^{3\pi i}$.
   (iii) Find all possible values of $(-1 + \sqrt{3}i)^{3/2}$.

3. Let $c = a + ib$ be a fixed complex number, where $c$ is not an integer. Find all possible values of $i^c$. What restriction must be placed on the constant $c$ so that the values of $|i^c|$ are all the same?

4. Show, by means of an example, that $\log(z_1z_2) \neq \log(z_1) + \log(z_2)$ in general.

5. (i) Suppose that $\alpha$ is a fixed real number. Show that $|z^\alpha| = e^{\alpha \ln(|z|)}$ for every nonzero complex number $z$.
   (ii) Suppose that $\theta_0$ is a fixed real number, and let $\log(z) = \ln|z| + i \arg(z)$, $\theta_0 \leq \arg(z) < \theta_0 + 2\pi$, $z \neq 0$.

   This gives a function $\mathbb{C} \setminus \{0\} \ni z \mapsto \log(z)$. Suppose now that $\alpha > 0$ is a fixed number, and define $f(z) := \begin{cases} z^\alpha = \exp(\alpha \log(z)), & \text{if } z \neq 0; \\ 0, & \text{if } z = 0. \end{cases}$

   Show that $f$ is continuous at the origin.

Definition. (i) Let $(x_0, y_0)$ be a fixed point in $\mathbb{R}^2$. Given $R > 0$, we define $B((x_0, y_0); R) := \{(x, y) \in \mathbb{R}^2 : (x-x_0)^2 + (y-y_0)^2 < R^2\}$.

(ii) Suppose that $H : B((x_0, y_0); R) \to \mathbb{R}$ is a real-valued function of two variables. We say that $H$ is continuous at the point $(x_0, y_0)$ if, for every pair of real sequences $\{x_n\}$ and $\{y_n\}$ such that $(x_n, y_n) \in B((x_0, y_0); R)$, $n \in \mathbb{N}$, $\lim_{n \to \infty} x_n = x_0$, and $\lim_{n \to \infty} y_n = y_0$, we have $\lim_{n \to \infty} H(x_n, y_n) = H(x_0, y_0)$.

6. Let $z_0 = x_0 + iy_0$ be a fixed complex number, and let $R > 0$ be fixed. Suppose that $f : D(z_0; R) \to \mathbb{C}$ is a function, and write $f(z) = u(x, y) + iv(x, y)$, $z = x + iy$. Note that $u$ and $v$ are real-valued functions defined on $B((x_0, y_0); R)$. Show that the following are equivalent:
   (i) $f$ is continuous at $z_0$.
   (ii) Both $u$ and $v$ are continuous at $(x_0, y_0)$.
7. Suppose $n$ is a fixed positive integer.

(i) Use Question 4 from Example Sheet 2 to find $\lim_{z \to 1} \frac{z^n - 1}{z - 1}$.

(ii) Let $\omega_0, \ldots, \omega_{n-1}$ denote the $n$ distinct $n$-th roots of unity; assume $\omega_0 = 1$. Let $d_k$ denote the distance between $\omega_k$ and $\omega_0$ for every $1 \leq k \leq n - 1$. Observe that

$$z^n - 1 = (z - \omega_0)(z - \omega_1) \cdots (z - \omega_{n-1}), \quad z \in \mathbb{C}.$$ 

(iii) Use (i) and (ii) to show that the product $d_1 d_2 \cdots d_{n-1} = n$.

(iv) Consider the $n - 1$ diagonals of a regular $n$-gon inscribed in a unit circle obtained by connecting one vertex with all the others. Show that the product of their lengths is $n$. (Use the foregoing result.)