1. Let \( a < b \) be real numbers, and suppose that \( G, H : [a, b] \to \mathbb{C} \) are continuous on \([a, b]\). Let \( \alpha \) be a fixed complex number. Verify the following statements:

(i) \( \int_{a}^{b} (G(t) + H(t)) \, dt = \int_{a}^{b} G(t) \, dt + \int_{a}^{b} H(t) \, dt \).

(ii) \( \int_{a}^{b} (\alpha G(t)) \, dt = \alpha \int_{a}^{b} G(t) \, dt \).

2. Suppose that \( \gamma \) is a smooth arc parametrized by \( z(t), a \leq t \leq b \). Let \( \phi : [c, d] \to [a, b] \) be a function satisfying the following conditions: (i) \( \phi \) is continuously differentiable on \([c, d]\), (ii) \( \phi'(s) > 0 \) for every \( c < s < d \), and (iii) \( \phi(c) = a \) and \( \phi(d) = b \).

(i) Verify that the function \( w : [c, d] \to \mathbb{C} \) given by \( w(s) := z(\phi(s)), c \leq s \leq d \), is continuously differentiable on \([c, d]\).

(ii) Verify that the function \( w \) also parametrizes \( \gamma \).

(iii) Let \( f : \gamma \to \mathbb{C} \) be continuous on \( \gamma \). Show that \( \int_{c}^{d} f(w(s))w'(s) \, ds = \int_{a}^{b} f(z(t))z'(t) \, dt \). (This shows that the line integral of \( f \) over \( \gamma \) is invariant under a smooth change of parameter.)

3. Compute each of the following integrals:

(i) \( \int_{\gamma} (z^2 - 3|z| + \text{Im}(z)) \, dz \), where \( \gamma \) is the quarter circle (of radius 2) centred at the origin and extending from 2 to 2i.

(ii) \( \int_{\gamma} (z - i) \, dz \), where \( \gamma \) is the parabolic segment determined by \( z(t) = t + it^2, -1 \leq t \leq 1 \).

(iii) \( \int_{\gamma} \cos z \, dz \), where \( \gamma \) is the line segment from \((-\pi/2) + i\) to \(\pi + i\).

The next two examples involve applications of the M-L Estimate.

4. Suppose \( \alpha > 1 \) is a fixed (real) number. Let \( \text{Log}(z) \) and \( z^\alpha \) denote the principal branches of the corresponding functions. Let \( C_R \) (\( R > 0 \)) denote the semicircle in (the closed right-half plane) \( \overline{H} \) which is centred at the origin and has radius \( R \). Show that \( \lim_{R \to \infty} \int_{C_R} \frac{\text{Log}(z)}{z^\alpha} \, dz = 0 \).

5. Let \( C_N \) denote the boundary of the square formed by the lines

\[
x = \pm(N + \frac{1}{2})\pi \quad \text{and} \quad y = \pm(N + \frac{1}{2})\pi,
\]

where \( N \) is a positive integer. Let the orientation of \( C_N \) be counterclockwise.

(a) Use Question 6(ii) from Example Sheet 4b to show that \( |\sin z| \geq |\sin x| \), where \( z = x + iy \).

(b) Recall from Question 6(iii) in Example Sheet 4b that \( |\sin z| \geq |\sinh y| \), \( z = x + iy \).

(c) Use (a) and (b) to show that \( |\sin z| \geq 1 \) when \( z \) lies on the vertical sides of the aforementioned square, whereas \( |\sin z| \geq \sinh(\pi/2) \) when \( z \) lies on the horizontal sides. Conclude that there is a constant \( A \), independent of \( N \), such that \( |\sin z| \geq A \) for all points \( z \) lying on the contour \( C_N \).
(d) Prove that

\[ \lim_{N \to \infty} \int_{C_N} \frac{dz}{z^2 \sin z} = 0. \]

6. Suppose \( f \) is a continuous real-valued function and \( |f(z)| \leq 1 \) for every \( z \in C(0; 1) \). Prove that

\[ \left| \int_{C(0;1)} f(z) \, dz \right| \leq 4. \]

(Suggestion: Show that \( \left| \int_{C(0;1)} f(z) \, dz \right| \leq \int_0^{2\pi} |\sin \sigma| \, d\sigma. \))