Example Sheet 8

1. Let $\sqrt{z}$ denote the principal branch of the multivalued function $z \mapsto z^{1/2}$, and define $g(z) := \frac{\exp(iz)}{\sqrt{z}}$.

(a) Let $\epsilon$ and $R$ be positive numbers with $\epsilon < R$. Consider the (simple closed) contour $C = C(\epsilon, R)$ comprising the following four pieces: the straight-line segment $L_1$ (along the real axis) from $z = \epsilon$ to $z = R$, followed by the quarter-circle $\Gamma_R$ (traversed counterclockwise) from $z = R$ to $z = iR$, followed by the line segment $L_2$ (along the imaginary axis) from $z = iR$ to $z = i\epsilon$, followed by the quarter-circle $\gamma_\epsilon$ (described clockwise) from $z = i\epsilon$ to $z = \epsilon$. Verify that

$$\int_{L_1} g(z) \, dz = \int_{-L_2} g(z) \, dz - \int_{\gamma_\epsilon} g(z) \, dz - \int_{\Gamma_R} g(z) \, dz.$$

(b) Parametrize $L_1$ and $-L_2$; use (a) to show that

$$R \int_{\epsilon}^{R} e^{it} \sqrt{t} \, dt = \int_{\epsilon}^{R} e^{-it} \sqrt{t} \, dt - \int_{\gamma_\epsilon} g(z) \, dz - \int_{\Gamma_R} g(z) \, dz.$$

(c) Show that

$$\lim_{\epsilon \to 0^+} \int_{\gamma_\epsilon} g(z) \, dz = 0 = \lim_{R \to \infty} \int_{\Gamma_R} g(z) \, dz.$$

(d) Show that

$$\int_{0}^{\infty} e^{it} \sqrt{t} \, dt = \lim_{R \to \infty} \int_{0}^{R} e^{it} \sqrt{t} \, dt = \frac{\sqrt{\pi}}{2} (1 + i).$$

(e) Deduce that

$$\int_{0}^{\infty} \frac{\cos t}{\sqrt{t}} \, dt = \int_{0}^{\infty} \frac{\sin t}{\sqrt{t}} \, dt = \frac{\sqrt{\pi}}{2}.$$

(The interpretation of each integral above is the same as in (d), i.e., $\lim_{R \to \infty} \int_{0}^{R} \cdots$)

(f) Use (e) to compute the Fresnel integrals.

2. Show that $|\exp(iz) - 1| \leq |z|$ for every $z$ whose imaginary part is nonnegative.

3. (i) Define $f(z) = \frac{e^{iz}}{z}$, $z \neq 0$, and let $0 < \rho < R$. Consider the contour $C$ consisting of the following pieces: the straight-line segment (along the real axis) from $z = \rho$ to $z = R$; followed by a semicircle $\Gamma$ from $z = R$ to $z = -R$, traversed counterclockwise; followed by the straight-line segment (along the real axis) from $z = -R$ to $z = -\rho$; and finally a semicircle $\gamma$ from $z = -\rho$ to $z = \rho$, traversed clockwise. Verify that $\int_{C} f(z) \, dz = 0$. 

(ii) Deduce from (i) that
\[ \int_{-\infty}^{\infty} \frac{e^{ix}}{x} \, dx + \int_{R} \frac{e^{iz}}{z} \, dz + \int_{-\infty}^{R} \frac{e^{-ix}}{-x} \, dx + \int_{-\gamma} \frac{e^{iz}}{z} \, dz = 0. \]

(iii) Use (ii) to show that
\[ 2i \int_{\rho} \sin x \frac{e^{ix}}{x} \, dx = -\int_{\Gamma} \frac{e^{iz}}{z} \, dz + \int_{-\gamma} \frac{e^{iz}}{z} \, dz. \]

(iv) Use Question 6 from Example Sheet 7 to show that
\[ \lim_{R \to \infty} \int_{\Gamma} \frac{e^{iz}}{z} \, dz = 0. \]

(v) Show that
\[ \int_{-\gamma} \frac{dz}{z} = i\pi. \]

(vi) Use (v) and Question 2 of the present example sheet to prove that
\[ \lim_{\rho \to 0^{+}} \int_{-\gamma} \frac{e^{iz}}{z} \, dz = i\pi. \]

(vii) Conclude that
\[ \int_{0}^{\infty} \frac{\sin x}{x} \, dx := \lim_{R \to \infty} \int_{\rho} \frac{\sin x}{x} \, dx = \frac{\pi}{2}. \]

4. Suppose \( C \) is a simple closed contour and \( D \) is the region enclosed by \( C \). Let \( \gamma_1 \) and \( \gamma_2 \) be two simple closed contours lying entirely in \( D \). Let \( D_{\gamma_1} \) and \( D_{\gamma_2} \) denote the regions enclosed by \( \gamma_1 \) and \( \gamma_2 \), respectively. Assume that \( \gamma_1 \cup D_{\gamma_1} \) and \( \gamma_2 \cup D_{\gamma_2} \) are disjoint (that is, the two sets do not intersect). Let \( f \) be a function which is analytic at every point on \( C \) and also at every point in \( D \setminus (D_{\gamma_1} \cup D_{\gamma_2}) \). Show that
\[ \int_{C} f(z) \, dz = \int_{\gamma_1} f(z) \, dz + \int_{\gamma_2} f(z) \, dz. \]
(Assume positive orientation for all three contours.)

5. Let \( C \) denote the boundary of the square whose sides lie along the lines \( x = \pm 2 \) and \( y = \pm 2 \), where \( C \) is taken to be positively oriented. Evaluate each of the following integrals:

(i) \[ \int_{C} \frac{z}{2z + 4} \, dz \]

(ii) \[ \int_{C} \frac{\cos z}{z(z^2 + 8)} \, dz \]

6. Let \( C \) denote the (positively oriented and once traversed) unit circle \( C(0; 1) \), and suppose that \( a \) is a fixed real number.

(i) Show that \[ \int_{C} \frac{e^{az}}{z} \, dz = 2\pi i. \]

(ii) Deduce that \[ \int_{0}^{\pi} e^{a \cos \theta} \cos(a \sin \theta) \, d\theta = \pi. \]
7. Suppose that $C$ is a simple closed (positively oriented) contour enclosing a region $D$. Let $f$ be analytic at every point in $C \cup D$, and let $a$ and $b$ be two points in $D$. Prove that
\[ \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)(z-b)} \, dz = \frac{f(a) - f(b)}{a-b}. \]

The following example is devoted to the computation of the Fourier Cosine Transform of the Poisson kernel, namely the function $t \mapsto (1 + t^2)^{-1}$.

8. (i) Suppose that $u$ is a fixed real number. Use the comparison theorem for infinite integrals to show that the integral $\int_{-\infty}^{\infty} \frac{\cos ut}{1 + t^2} \, dt$ is (absolutely) convergent.

(ii) Let $g(u) := \int_{-\infty}^{\infty} \frac{\cos ut}{1 + t^2} \, dt$, $u \in \mathbb{R}$. In view of (i), this integral can be evaluated as a Cauchy Principal Value; that is,
\[ g(u) = \lim_{R \to \infty} \int_{-R}^{R} \frac{\cos ut}{1 + t^2} \, dt, \quad u \in \mathbb{R}. \]

This will be our modus operandi to compute $g$ in the upcoming steps.

(iii) Compute $g(0)$.

(iv) Verify that $g(u) = \lim_{R \to \infty} G_R(u)$, where $G_R(u) := \int_{-R}^{R} \frac{e^{-iu t}}{1 + t^2} \, dt$, $u \in \mathbb{R}$.

(v) Suppose that $u < 0$. Use contour integration to show that $g(u) = \pi e^u = \pi e^{-|u|}$. Specifically, consider the function $\frac{e^{-iu z}}{1 + z^2}$ and the contour consisting of the following two pieces: the straight-line segment $L_R$ (along the real axis) from $z = -R$ to $z = R$, followed by the semicircle $\Gamma_R$ from $z = R$ to $z = -R$, traversed counterclockwise.

(vi) Use (ii) to show that $g$ is an even function, that is, $g(-u) = g(u)$ for every real number $u$.

(vii) Use (vi), (v), and (iii) to conclude that $g(u) = \pi e^{-|u|}$ for every real number $u$. 