Example Sheet 9

1. Use induction to complete the proof of (the general case of) Cauchy’s Integral Formula for Derivatives.

2. Suppose that $f$ is analytic in a region $D$. It was proved in lecture that, if $f'(z) = 0$ for every $z \in D$, then $f$ is a constant throughout $D$. Generalize this to the following: if $f$ is analytic in a region $D$ and $f^{(n)}(z) = 0$ for some $n \in \mathbb{N}$ and every $z \in D$, then $f$ must be a polynomial of degree at most $n - 1$. (Suggestion: Use induction on $n$.)

3. (i) Suppose that $f$ is analytic at every point of the closed disc $\overline{D}(a; R)$, where $a \in \mathbb{C}$ and $R > 0$. Prove Cauchy’s inequality:

$$|f^{(n)}(a)| \leq \frac{M n!}{R^n},$$

where $M := \sup\{|f(z)| : z \in C(a; R)\}$.

(ii) Use (i) to prove Liouville’s Theorem.

(iii) Suppose that $f$ is entire and $|f(z)| \leq A + B|z|^{3/2}$ for every $z \in \mathbb{C}$, where $A$ and $B$ are real constants. Show that $f$ is a linear polynomial (i.e., a polynomial of degree at most 1).

(iv) Suppose that $f$ is an entire function. Assume that there exist positive numbers $A$ and $T$ and a positive integer $m$ such that $|f(z)| \leq A|z|^m$ whenever $|z| > T$. Show that $f$ is a polynomial of degree at most $m$.

4. Suppose that $P(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial such that $|P(z)| \leq 1$ for every $z \in \overline{D}(0; 1)$. Show that $|a_k| \leq 1$ for every $0 \leq k \leq n$.

5. Suppose that $f$ is entire and $|f'(z)| \leq |z|$ for every $z \in \mathbb{C}$. Show that $f(z) = a + cz^2$, where $a, c \in \mathbb{C}$ and $|c| \leq 1/2$.

6. Prove the following special case of Morera’s Theorem: Let $f$ be continuous at every point in an open convex set $S$. The following are equivalent:

(a) $\int_{C} f(z) \, dz = 0$ for every simple closed contour contained in $S$.

(b) If $\Delta$ is any triangle contained in $S$ and $\partial \Delta$ denotes its boundary, then $\int_{\partial \Delta} f(z) \, dz = 0$.

(c) $f$ is analytic throughout $S$.

(See Question 2 in Example Sheet 7.)

7. Suppose that $f$ is entire and that for every $z \in \mathbb{C}$, either $|f(z)| \leq 1$ or $|f'(z)| \leq 1$.

(i) Use a line integral to show that $|f(z)| \leq A + |z|$, where $A = \max\{1, |f(0)|\}$.

(ii) Deduce that $f$ must be a linear polynomial.