- Suppose that S is a nonempty subset of the real line.
  (i) A real number a is said to be a lower bound for S if $a \leq x$ for every $x \in S$.
  (ii) A real number b is said to be an upper bound for S if $x \leq b$ for every $x \in S$.
  (iii) A real number α is called the infimum or the greatest lower bound of S if α is a lower bound for S, and, if a is any lower bound for S, then $a \leq \alpha$.
  (iv) A real number β is called the supremum or the least upper bound of S if β is an upper bound for S, and, if b is any upper bound for S, then $\beta \leq b$.

- **Least Upper Bound Axiom:** Every nonempty subset of the real line which is bounded above has a least upper bound in $\mathbf{R}$.

- **Archimedean Property:** Given a pair of positive numbers a and b, there is a positive integer n such that $na > b$.

- Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers.
  (i) Let L be a real number. We say that $\lim\limits_{n\to\infty} a_n = L$ if, given $\epsilon > 0$, there is a positive integer N such that $|a_n - L| < \epsilon$ for every $n \geq N$.
  (ii) We say that $\lim\limits_{n\to\infty} a_n = +\infty$ if, given $T > 0$, there is a positive integer N such that $a_n > T$ for every $n \geq N$.

- A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be Cauchy if, given $\epsilon > 0$, there is a positive integer N such that $|a_n - a_m| < \epsilon$ whenever $m, n \geq N$.

- Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers.
  (i) If $\{a_n\}_{n=1}^{\infty}$ is not bounded above, then we define $\limsup\limits_{n\to\infty} a_n = +\infty$.
  (ii) Suppose that $\{a_n\}_{n=1}^{\infty}$ is bounded above. Let

$$M_n := \sup\{a_k : k \geq n\}, \qquad n \in \mathbf{N}.$$

  Define

$$\limsup_{n\to\infty} a_n = \begin{cases} -\infty, & \text{if } \{M_n\}_{n=1}^{\infty} \text{ is not bounded below;} \\ \inf\{M_n : n \in \mathbf{N}\}, & \text{otherwise.} \end{cases}$$

- Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers.
  (i) If $\{a_n\}_{n=1}^{\infty}$ is not bounded below, then we define $\liminf\limits_{n\to\infty} a_n = -\infty$.
  (ii) Suppose that $\{a_n\}_{n=1}^{\infty}$ is bounded below. Let

$$m_n := \inf\{a_k : k \geq n\}, \qquad n \in \mathbf{N}.$$

  Define

$$\liminf_{n\to\infty} a_n = \begin{cases} +\infty, & \text{if } \{m_n\}_{n=1}^{\infty} \text{ is not bounded above;} \\ \sup\{m_n : n \in \mathbf{N}\}, & \text{otherwise.} \end{cases}$$

- Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers, and let $m, M \in \mathbf{R}$.
  (i) $M = \limsup\limits_{n\to\infty} a_n$ if and only if, given $\epsilon > 0$, $a_n < M + \epsilon$ for all but at most finitely many values of n, and $a_n > M - \epsilon$ for infinitely many values of n.

(ii) $m = \liminf_{n\to\infty} a_n$ if and only if, given $\epsilon > 0$, $a_n > m - \epsilon$ for all but at most finitely many values of n, and $a_n < m + \epsilon$ for infinitely many values of n.

• **Bolzano–Weierstrass Theorem:** Every bounded sequence of real numbers has a convergent subsequence.

• **Squeeze Theorem/Sandwich Principle for sequences:** Suppose that $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{c_n\}_{n=1}^{\infty}$ are sequences satisfying the following condition: there exists a positive integer n_0 such that $a_n \leq b_n \leq c_n$ for every $n \geq n_0$. If $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, then, $\lim_{n\to\infty} b_n = L$.