

Let  $a$  and  $b$  be real numbers with  $a < b$ .

- A partition of  $[a, b]$  is a finite sequence of numbers  $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ .
- Let  $P$  and  $Q$  be partitions of  $[a, b]$ . We say that  $Q$  is a refinement of  $P$  if every partition point of  $P$  is also a partition point of  $Q$ .
- Let  $f : [a, b] \rightarrow \mathbf{R}$  be a bounded function, and let  $P = \{x_0, \cdots, x_n\}$  be a partition of  $[a, b]$ . A Riemann sum associated to  $P$  and  $f$  is a sum of the form

$$S(P, f) = \sum_{k=1}^n f(\xi_k)(x_k - x_{k-1}),$$

where  $\xi_k \in [x_{k-1}, x_k]$  for every  $1 \leq k \leq n$ .

- Let  $f : [a, b] \rightarrow \mathbf{R}$  be a bounded function. We say that  $f$  is Riemann integrable on  $[a, b]$  if there is a number  $\mathcal{I}$  satisfying the following condition: given  $\epsilon > 0$ , there is a partition  $P_\epsilon$  of  $[a, b]$  such that  $|S(P, f) - \mathcal{I}| < \epsilon$  for every  $P$  which refines  $P_\epsilon$ , and every Riemann sum  $S(P, f)$  associated to  $P$  and  $f$ .
- Suppose that  $f$  is a bounded function defined on the interval  $J := [\alpha, \beta]$ . The oscillation of  $f$  over the interval  $J$  is defined as follows:

$$\omega(f, [\alpha, \beta]) := \sup\{|f(u) - f(v)| : \alpha \leq u, v \leq \beta\}.$$

- Let  $D$  be a nonempty subset of the real line. For each positive integer  $n$ , let  $f_n : D \rightarrow \mathbf{R}$  be a function. Let  $f : D \rightarrow \mathbf{R}$  be a function. We say that the sequence  $\{f_n\}_{n=1}^\infty$  converges to  $f$  pointwise on  $D$  if  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  for every  $x \in D$ .
- Let  $D$  be a nonempty subset of the real line, and let  $\{f_n\}_{n=1}^\infty$  be a sequence of functions defined on  $D$ . Let  $f : D \rightarrow \mathbf{R}$  be a function. We say that  $\{f_n\}_{n=1}^\infty$  converges to  $f$  uniformly on  $D$  if the following holds: given  $\epsilon > 0$ , there is a positive integer  $N$  such that  $|f_n(x) - f(x)| < \epsilon$  for every  $n \geq N$  and every  $x \in D$ .

**Rolle's Theorem:** Let  $a$  and  $b$  be real numbers with  $a < b$ . Let  $H : [a, b] \rightarrow \mathbf{R}$  be a function satisfying the following conditions: (i)  $H$  is continuous on  $[a, b]$ , (ii)  $H$  is differentiable on  $(a, b)$ , and (iii)  $H(a) = H(b)$ . Then there is a point  $c \in (a, b)$  such that  $H'(c) = 0$ .

**Cauchy's Mean Value Theorem:** Let  $a$  and  $b$  be real numbers with  $a < b$ . Let  $f, g : [a, b] \rightarrow \mathbf{R}$  be a pair of functions satisfying the following conditions: (i)  $f$  and  $g$  are continuous on  $[a, b]$ , and (ii)  $f$  and  $g$  are differentiable on  $(a, b)$ . Then there is a point  $c \in (a, b)$  such that  $g'(c)[f(b) - f(a)] = f'(c)[g(b) - g(a)]$ .

**Lagrange's Mean Value Theorem:** Let  $a$  and  $b$  be real numbers with  $a < b$ . If  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then there is a point  $c \in (a, b)$  such that  $f(b) - f(a) = f'(c)(b - a)$ .

**Taylor's Remainder Theorem:** Let  $a$  and  $b$  be real numbers with  $a < b$ . Suppose that  $n$  is a natural number, that  $f$  and its derivatives  $f', f'', \dots, f^{(n-1)}$  are defined and continuous on the interval  $[a, b]$ , and that  $f^{(n)}$  exists in  $(a, b)$ . If  $\alpha, \beta \in [a, b]$ , then there is a number  $\gamma$  between  $\alpha$  and  $\beta$  such that

$$f(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (\beta - \alpha)^k + \frac{f^{(n)}(\gamma)}{n!} (\beta - \alpha)^n.$$

**Differentiation Theorem for Inverse Functions:** Let  $f$  be a one-to-one continuous function on an open interval  $I$ . Suppose that  $f$  is differentiable at  $x_0 \in I$ , and that  $f'(x_0) \neq 0$ . Then  $f^{-1}$  is differentiable at  $y_0 = f(x_0)$ , and  $(f^{-1})'(y_0) = 1/f'(x_0)$ .

**Cauchy Criterion for Riemann Integrals:** Let  $a$  and  $b$  be real numbers with  $a < b$ , and let  $f : [a, b] \rightarrow \mathbf{R}$  be a bounded function. Then  $f$  is Riemann integrable on  $[a, b]$  if and only if, given  $\epsilon > 0$ , there is a partition  $P_\epsilon$  of  $[a, b]$  such that  $|S(P, f) - S(Q, f)| < \epsilon$  for every  $P$  and  $Q$  refining  $P_\epsilon$ , for every  $S(P, f)$  associated to  $P$  and  $f$ , and every  $S(Q, f)$  associated to  $Q$  and  $f$ .

**Riemann Criterion for Riemann Integrals:** Let  $a$  and  $b$  be real numbers with  $a < b$ , and let  $f : [a, b] \rightarrow \mathbf{R}$  be a bounded function. The following are equivalent:

- (i)  $f$  is Riemann integrable on  $[a, b]$ .
- (ii) Given  $\epsilon > 0$ , there is a partition  $P_\epsilon$  of  $[a, b]$  such that, if  $P = \{x_0, \dots, x_n\}$  is any refinement of  $P_\epsilon$ , then

$$\sum_{k=1}^n \omega(f, [x_{k-1}, x_k])(x_k - x_{k-1}) < \epsilon.$$

**Fundamental Theorem of Calculus:** Let  $a$  and  $b$  be real numbers with  $a < b$ .

- (i) Let  $f$  be continuous on  $[a, b]$ , and define

$$F(x) := \int_a^x f(t) dt, \quad a \leq x \leq b.$$

Then  $F'_+(a) = f(a)$ ,  $F'_-(b) = f(b)$ , and  $F'(x) = f(x)$  for every  $a < x < b$ .

- (ii) If  $g$  is differentiable on  $[a, b]$ , and if  $g'$  is Riemann integrable on  $[a, b]$ , then

$$\int_a^b g' = g(b) - g(a).$$