Let $a$ and $b$ be real numbers with $a < b$.

- A partition of $[a, b]$ is a finite sequence of numbers $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$.
- Let $P$ and $Q$ be partitions of $[a, b]$. We say that $Q$ is a refinement of $P$ if every partition point of $P$ is also a partition point of $Q$.
- Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and let $P = \{x_0, \cdots, x_n\}$ be a partition of $[a, b]$. A Riemann sum associated to $P$ and $f$ is a sum of the form

$$S(P, f) = \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}),$$

where $\xi_k \in [x_{k-1}, x_k]$ for every $1 \leq k \leq n$.
- Let $f : [a, b] \to \mathbb{R}$ be a bounded function. We say that $f$ is Riemann integrable on $[a, b]$ if there is a number $I$ satisfying the following condition: given $\epsilon > 0$, there is a partition $P_\epsilon$ of $[a, b]$ such that $|S(P_\epsilon, f) - I| < \epsilon$ for every $P_\epsilon$ which refines $P$, and every Riemann sum $S(P, f)$ associated to $P$ and $f$.
- Suppose that $f$ is a bounded function defined on the interval $J := [\alpha, \beta]$. The oscillation of $f$ over the interval $J$ is defined as follows:

$$\omega(f, [\alpha, \beta]) := \sup\{|f(u) - f(v)| : \alpha \leq u, v \leq \beta\}.$$

- Let $D$ be a nonempty subset of the real line. For each positive integer $n$, let $f_n : D \to \mathbb{R}$ be a function. Let $f : D \to \mathbb{R}$ be a function. We say that the sequence $\{f_n\}_{n=1}^\infty$ converges to $f$ pointwise on $D$ if $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in D$.
- Let $D$ be a nonempty subset of the real line, and let $\{f_n\}_{n=1}^\infty$ be a sequence of functions defined on $D$. Let $f : D \to \mathbb{R}$ be a function. We say that $\{f_n\}_{n=1}^\infty$ converges to $f$ uniformly on $D$ if the following holds: given $\epsilon > 0$, there is a positive integer $N$ such that $|f_n(x) - f(x)| < \epsilon$ for every $n \geq N$ and every $x \in D$.

**Rolle’s Theorem**: Let $a$ and $b$ be real numbers with $a < b$. Let $H : [a, b] \to \mathbb{R}$ be a function satisfying the following conditions: (i) $H$ is continuous on $[a, b]$, (ii) $H$ is differentiable on $(a, b)$, and (iii) $H(a) = H(b)$. Then there is a point $c \in (a, b)$ such that $H'(c) = 0$.

**Cauchy’s Mean Value Theorem**: Let $a$ and $b$ be real numbers with $a < b$. Let $f, g : [a, b] \to \mathbb{R}$ be a pair of functions satisfying the following conditions: (i) $f$ and $g$ are continuous on $[a, b]$, and (ii) $f$ and $g$ are differentiable on $(a, b)$. Then there is a point $c \in (a, b)$ such that $g'(c)[f(b) - f(a)] = f'(c)[g(b) - g(a)]$.

**Lagrange’s Mean Value Theorem**: Let $a$ and $b$ be real numbers with $a < b$. If $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then there is a point $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.

**Taylor’s Remainder Theorem**: Let $a$ and $b$ be real numbers with $a < b$. Suppose that $n$ is a natural number, that $f$ and its derivatives $f', f''$, \ldots, $f^{(n-1)}$ are defined and continuous on the interval $[a, b]$, and that $f^{(n)}$ exists in $(a, b)$. If $\alpha, \beta \in [a, b]$, then there is a number $\gamma$ between $\alpha$ and $\beta$ such that

$$f(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k + \frac{f^{(n)}(\gamma)}{n!}(\beta - \alpha)^n.$$

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**Differentiation Theorem for Inverse Functions:** Let $f$ be a one-to-one continuous function on an open interval $I$. Suppose that $f$ is differentiable at $x_0 \in I$, and that $f'(x_0) \neq 0$. Then $f^{-1}$ is differentiable at $y_0 = f(x_0)$, and $(f^{-1})'(y_0) = 1/f'(x_0)$.

**Cauchy Criterion for Riemann Integrals:** Let $a$ and $b$ be real numbers with $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then $f$ is Riemann integrable on $[a, b]$ if and only if, given $\epsilon > 0$, there is a partition $P_\epsilon$ of $[a, b]$ such that $|S(P, f) - S(Q, f)| < \epsilon$ for every $P$ and $Q$ refining $P_\epsilon$, for every $S(P, f)$ associated to $P$ and $f$, and every $S(Q, f)$ associated to $Q$ and $f$.

**Riemann Criterion for Riemann Integrals:** Let $a$ and $b$ be real numbers with $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a bounded function. The following are equivalent:

(i) $f$ is Riemann integrable on $[a, b]$.

(ii) Given $\epsilon > 0$, there is a partition $P_\epsilon$ of $[a, b]$ such that, if $P = \{x_0, \cdots, x_n\}$ is any refinement of $P_\epsilon$, then

$$
\sum_{k=1}^{n} \omega(f, [x_{k-1}, x_k])(x_k - x_{k-1}) < \epsilon.
$$

**Fundamental Theorem of Calculus:** Let $a$ and $b$ be real numbers with $a < b$.

(i) Let $f$ be continuous on $[a, b]$, and define

$$
F(x) := \int_{a}^{x} f(t) \, dt, \quad a \leq x \leq b.
$$

Then $F'_+(a) = f(a)$, $F'_-(b) = f(b)$, and $F'(x) = f(x)$ for every $a < x < b$.

(ii) If $g$ is differentiable on $[a, b]$, and if $g'$ is Riemann integrable on $[a, b]$, then

$$
\int_{a}^{b} g' = g(b) - g(a).
$$