MATH 409-08a, Exam 1

SOLUTIONS TO SELECTED QUESTIONS

7. (i) As S is bounded, there exist numbers A and B such that $A \le s \le B$ for every s in S. Likewise there exist numbers A' and B' such that $A' \le t \le B'$ for every $t \in T$. Therefore

$$A + A' \le s + t \le B + B', \qquad \forall\, s \in S,\, t \in T,$$

proving that the set $S + T$ is bounded below by $A + A'$ and bounded above by $B + B'$.

(ii) Let $\alpha := \sup(S)$ and $\beta := \sup(T)$. If $s \in S$ and $t \in T$, then $s \le \alpha$ and $t \in \beta$, so $s + t \le \alpha + \beta$. As this is true for every $s \in S$ and $t \in T$, we conclude that $\alpha + \beta$ is an upper bound for $S + T$. Next we show that no number smaller than $\alpha + \beta$ can be an upper bound for $S + T$. Consider such a number $\alpha + \beta - \epsilon$, $\epsilon > 0$. As $\alpha - \epsilon/2 < \alpha$, and the latter is the least upper bound of S, there exists some $s_0 \in S$ such that $\alpha - \epsilon/2 < s_0$. Similarly there is some $t_0 \in T$ such that $\beta - \epsilon/2 < t_0$. Therefore

$$\alpha + \beta - \epsilon = (\alpha - \epsilon/2) + (\beta - \epsilon/2) < s_0 + t_0.$$

As $s_0 + t_0 \in S + T$, the inequality above demonstrates that $\alpha + \beta - \epsilon$ cannot be an upper bound for $S + T$.

8. In each of the examples below, verifications are left to the reader.

(i) Let $a_n = (-1)^n$, $n \in \mathbf{N}$. Then $\{a_n\}_{n=1}^{\infty}$ is divergent, hence not Cauchy. However $|a_n| = 1$ for every n, hence $\lim\limits_{n\to\infty} |a_n| = 1$; in particular the sequence $\{|a_n|\}_{n=1}^{\infty}$ is Cauchy.

(ii) $a_n = n$, $n \in \mathbf{N}$. Verify that $m_n = n$ for each n, hence the result.

(iii) Choose

$$a_n = \begin{cases} -1, & \text{if } n \text{ is odd;} \\ 0, & \text{otherwise.} \end{cases}$$

Then $\sup\{a_k : k \ge n\} = 0$ for every n, so $\limsup\limits_{n\to\infty} a_n = 0$. Now

$$a_n^2 = \begin{cases} 1, & \text{if } n \text{ is odd;} \\ 0 & \text{otherwise.} \end{cases}$$

Here $\sup\{a_k^2 : k \ge n\} = 1$ for every n, so $\limsup\limits_{n\to\infty} a_n^2 = 1$.

(iv) $a_n = (n+2)^2$, $b_n = n^2$, $n \in \mathbf{N}$.

9. Suppose that statement (a) holds. We show that (b) follows. Let $\epsilon > 0$ be given. Assumption (a) provides a positive integer N such that $|a_n| = |a_n - 0| < \epsilon$ for every $n \ge N$. Thus $|\,|a_n| - 0| < \epsilon$ for every $n \ge N$, and (b) is proven.

Assume now that (b) holds; we show that (c) follows. Let $T > 0$ be given. As (b) holds, there is a positive integer M such that $|a_n| < 1/T$ for every $n \ge M$. Therefore $1/|a_n| > T$ for every such n, and this proves (c).

Finally, assume that (c) holds. Given $\epsilon > 0$, this assumption gives a positive integer K such that $1/|a_n| > 1/\epsilon$ for every $n \ge K$. It follows that $|a_n| < \epsilon$, and (a) is proven.

10. Suppose firstly that (a) holds, that is, $\lim_{n\to\infty} a_n = 0$. A theorem proved in class asserts the following: if $\lim_{n\to\infty} a_n = L$, then $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = L$. Thus (b) follows.

Conversely, suppose that $\limsup_{n\to\infty} a_n = 0$. Let $M_n := \sup\{a_k : k \geq n\}$, and recall that $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} M_n$. Thus $\lim_{n\to\infty} M_n = 0$ in our case. As each a_n is given to be nonnegative, we have

$$0 \leq a_n \leq M_n, \qquad \forall\, n \in \mathbf{N},$$

whence the required result obtains via the Sandwich Principle.

11. (i) As $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are bounded, so is the sequence $\{s_n + t_n\}_{n=1}^{\infty}$. So each of these three sequences admits a finite limit inferior. Now let

$$m_n(s) := \inf\{s_k : k \geq n\},\ m_n(t) := \inf\{t_k : k \geq n\},\ \text{and } m_n(s{+}t) := \inf\{s_k{+}t_k : k \geq n\}, \quad n \in \mathbf{N}.$$

In view of the remark above, each of the sequences $\{m_n(s)\}_{n=1}^{\infty}$, $\{m_n(t)\}_{n=1}^{\infty}$, and $\{m_n(s+t)\}_{n=1}^{\infty}$ converges to a finite limit; in fact,

$$\liminf_{n\to\infty} s_n = \lim_{n\to\infty} m_n(s), \quad \liminf_{n\to\infty} t_n = \lim_{n\to\infty} m_n(t), \quad \text{and} \quad \liminf_{n\to\infty}(s_n + t_n) = \lim_{n\to\infty} m_n(s+t). \tag{1}$$

Now, if $n \in \mathbf{N}$ and $k \geq n$, then $s_k + t_k \geq m_n(s) + m_n(t)$, so the latter number is a lower bound for the collection of numbers $\{s_k + t_k : k \geq n\}$. Consequently,

$$m_n(s+t) = \text{glb}\{s_k + t_k : k \geq n\} \geq m_n(s) + m_n(t).$$

Letting n approach infinity and invoking (1) yields the desired result.

(ii) Let $s_n := (-1)^n$ and $t_n := (-1)^{n-1}$, $n \in \mathbf{N}$.

(iii) Choose $s_n = 1 = t_n$ for every n.