MATH 409-08a, Exam 2

SOLUTIONS TO SELECTED QUESTIONS

6. (i) $(0,1)\cup(3,4)$; (ii) $[0,1)$; (iii) $[1,2]\cup[3,4]$; (iv) $(0,1)$; (v) $[0,\infty)$; (vi) $f(x)=1$ if x is rational, $=-1$ if x is irrational; (vii) $f(x)=x^2$; (viii) $f(x)=|x-1|+|x-2|$

7. Done in class.

8. Suppose that F is closed, and let p be a real number such that $(p-\epsilon,p+\epsilon)\cap F\neq\emptyset$ for every $\epsilon>0$. We must show that $p\in F$. If p does not belong to F, then it must belong to F^c. As F is closed, its complement is open, so there is some $r>0$ such that $(p-r,p+r)\subseteq F^c$. But this means that $(p-r,p+r)\cap F=\emptyset$, a contradiction. Hence $p\in F$.

Conversely, assume that condition (b) holds, that is, if p is a number such that $(p-\epsilon,p+\epsilon)\cap F\neq\emptyset$ for every $\epsilon>0$, then $p\in F$. We must show that F is closed, that is, we must show that F^c is open. If F^c is empty, there is nothing to prove, so let us assume that it is not empty. Let $g\in F^c$. We claim that there is some $R>0$ such that $(g-R,g+R)\subseteq F^c$. If no such R exists, it means that $(g-\epsilon,g+\epsilon)$ intersects F for every $\epsilon>0$. So $g\in F$ by assumption. But this is a contradiction (because $g\in F^c$), and the proof is complete.

9.

$$\begin{aligned}\lim_{x\to1}\frac{F(x)-F(1)}{x-1} &= \lim_{x\to1}\frac{xf(x)-1}{x-1}\\ &= \lim_{x\to1}\frac{xf(x)-x+x-1}{x-1}\\ &= \lim_{x\to1}\left[x\,\frac{f(x)-1}{x-1}+\frac{x-1}{x-1}\right]. \qquad (1)\end{aligned}$$

As $\lim_{x\to1}x=1$, $\lim_{x\to1}\frac{f(x)-1}{x-1}=f'(1)=1$, and $\lim_{x\to1}\frac{x-1}{x-1}=1$, we find from (1) that

$$\lim_{x\to1}\frac{F(x)-F(1)}{x-1}=(1)(1)+1=2.$$

Thus $F'(1)=2$.

10. See Quiz 9.