Assignment 10

1. Let $a < b$ be a pair of real numbers, and let $f : [a, b] \to \mathbb{R}$ be a bounded function. Prove that the Riemann criterion can be formulated in the following (equivalent) way: $f$ is Riemann integrable on $[a, b]$ if and only if, given $\epsilon > 0$, there is a partition $P_\epsilon := \{a = t_0 \leq t_1 \leq \cdots \leq t_r = b\}$ such that

$$\sum_{k=1}^{r} \omega(f, [t_{k-1}, t_k])(t_k - t_{k-1}) < \epsilon.$$

2. (i) Let $a$ and $b$ be real numbers with $a < b$. Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded nondecreasing function. Use the form of the Riemann Criterion given in the preceding question to prove that $f$ is Riemann integrable on $[a, b]$.

(ii) Deduce from (i) that a bounded nonincreasing function on $[a, b]$ is also Riemann integrable on that interval.

3. Let

$$f(x) := \begin{cases} 
0, & \text{if } x = 0; \\
\sin(1/x), & \text{if } 0 < x \leq 1.
\end{cases}$$

Use the form of the Riemann Criterion given in Question 1 to prove that $f$ is Riemann integrable on $[0, 1]$.

The following question is taken from the book Basic Analysis: Japanese Grade 11:

4. Find a function $f$ and a constant $a$ such that

$$\int_{1}^{x} f(t) \, dt = x^3 + ax - 5.$$

5. From the text: 33.8(b), 33.14, 34.5, 34.6, 34.12