Assignment 2

1. (i) Determine
$$\lim_{n\to\infty} \frac{2n^2+n+1}{n^2-n+2}.$$
(ii) Give a formal proof to justify your assertion in (i).

2. This was discussed in lecture. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers, and let L be a fixed real number. Verify that the following statements are equivalent:
(a) The sequence $\{a_n\}_{n=1}^{\infty}$ does *not* converge to L.
(b) There exists a positive number ϵ_0, and a sequence $\{n_k : k \in \mathbf{N}\}$ of positive integers, such that $n_k < n_{k+1}$ for every k, and $|a_{n_k} - L| \geq \epsilon_0$ for every k.
(c) There exists a positive number ϵ_0 such that $|a_n - L| \geq \epsilon_0$ for infinitely many values of n.

3. Let $a_n := (-1)^n$, $n \in \mathbf{N}$. Use the preceding result to show that $\{a_n\}_{n=1}^{\infty}$ does not converge to any real number.

4. (i) Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers, and that $\lim_{n\to\infty} a_n = L$. Prove that $\lim_{n\to\infty} |a_n| = |L|$.
(ii) Give an example of a sequence $\{a_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} |a_n|$ exists, but $\lim_{n\to\infty} a_n$ does not exist.

5. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers. Prove that the following statements are equivalent:
(a) $\lim_{n\to\infty} a_n = 0$.
(b) $\lim_{n\to\infty} |a_n| = 0$.
(c) $\lim_{n\to\infty} a_n^2 = 0$.

6. (i) Suppose that $\{c_n\}_{n=1}^{\infty}$ is a sequence of real numbers satisfying the following condition: there exists a positive integer n_0 such that $c_n \geq 0$ for every $n \geq n_0$. Assume further that $\lim_{n\to\infty} c_n = L$. Prove that $L \geq 0$. (Suppose that $L < 0$ and derive a contradiction.)
(ii) Give an example of a sequence of positive numbers which converges to zero.
(iii) Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences satisfying the following condition: there exists a positive integer n_0 such that $a_n \geq b_n$ for every $n \geq n_0$. Assume further that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Use (i) to show that $A \geq B$.

7. Questions from the text:
7.1(c), (d); 7.2 for these sequences
7.3(e), (g), (h), (i), (m), (p)
7.4, 7.5(a), 8.8(a)
8.4, 8.7(b)