Assignment 3

1. Suppose that $\lim_{n\to\infty} a_n = +\infty$. Let $\{b_n\}_{n=1}^{\infty}$ be a sequence satisfying the following condition: there is a positive number λ and a positive integer n_0 such that $b_n \geq \lambda$ for every $n \geq n_0$. Show that $\lim_{n\to\infty} a_n b_n = +\infty$.

2. Give examples of sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ as directed below:

(i) $\lim_{n\to\infty} a_n = +\infty = \lim_{n\to\infty} b_n$, and $\lim_{n\to\infty} (a_n - b_n) = 0$.
(ii) $\lim_{n\to\infty} a_n = +\infty = \lim_{n\to\infty} b_n$, and $\lim_{n\to\infty} (a_n - b_n) = 2$.
(iii) $\lim_{n\to\infty} a_n = +\infty = \lim_{n\to\infty} b_n$, and $\lim_{n\to\infty} (a_n - b_n) = +\infty$.
(iv) $\lim_{n\to\infty} a_n = +\infty = \lim_{n\to\infty} b_n$, and $\lim_{n\to\infty} (a_n - b_n) = -\infty$.

3. Give examples of sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ as directed below:

(i) $\lim_{n\to\infty} a_n = +\infty$, $\lim_{n\to\infty} b_n = 0$, and $\lim_{n\to\infty} a_n b_n = 3$.
(ii) $\lim_{n\to\infty} a_n = +\infty$, $\lim_{n\to\infty} b_n = 0$, and $\lim_{n\to\infty} a_n b_n = 0$.
(iii) $\lim_{n\to\infty} a_n = +\infty$, $\lim_{n\to\infty} b_n = 0$, and $\lim_{n\to\infty} a_n b_n = +\infty$.
(iv) $\lim_{n\to\infty} a_n = +\infty$, $\lim_{n\to\infty} b_n = 0$, and $\lim_{n\to\infty} a_n b_n = -\infty$.

4. Give a formal proof for the following:

$$\lim_{n\to\infty} \frac{2n-1}{\sqrt{n}+1} = +\infty.$$

5. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a given sequence, and let $\{a_{n_k} : k \in \mathbf{N}\}$ be a subsequence of $\{a_n\}_{n=1}^{\infty}$. Prove the following statements:

(i) If $\lim_{n\to\infty} a_n = L$, then $\lim_{k\to\infty} a_{n_k} = L$.
(ii) If $\lim_{n\to\infty} a_n = -\infty$, then $\lim_{k\to\infty} a_{n_k} = -\infty$.

6. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence defined recursively as follows:

$$a_1 = 1, \qquad a_{n+1} := \sqrt{1 + a_n}, \quad n \geq 1.$$

(i) Use induction to show that $\{a_n\}_{n=1}^{\infty}$ is increasing.
(ii) Use induction to show that $\{a_n\}_{n=1}^{\infty}$ is bounded above by 2.
(iii) Deduce that $\{a_n\}_{n=1}^{\infty}$ is convergent.
(iv) Determine the value of $\lim_{n\to\infty} a_n$.

7. From the text:

9.11, 9.12, 9.14, 9.15, 9.18, 10.8, 10.9, 10.10