1. Suppose that \( \{a_n\}_{n=1}^{\infty} \) is a sequence of real numbers. Prove that the following are equivalent:
   (a) \( \lim \sup_{n \to \infty} a_n = +\infty \).
   (b) There exists a subsequence of \( \{a_n\}_{n=1}^{\infty} \), say \( \{a_{n_k}\}_{k=1}^{\infty} \), such that \( \lim_{k \to \infty} a_{n_k} = +\infty \).

2. Suppose that \( \{a_n\}_{n=1}^{\infty} \) is a sequence of real numbers. Prove that the following are equivalent:
   (a) \( \lim_{n \to \infty} a_n = +\infty \).
   (b) \( \liminf_{n \to \infty} a_n = \limsup_{n \to \infty} a_n = +\infty \).

3. Suppose that \( \{a_n\}_{n=1}^{\infty} \) is a bounded sequence, and let \( m = \liminf_{n \to \infty} a_n \). Show that there is a subsequence of \( \{a_n\}_{n=1}^{\infty} \) which converges to \( m \).

Let \( \{a_n\}_{n=1}^{\infty} \) be a sequence of real numbers. A real number \( \lambda \) is said to be a subsequential limit of \( \{a_n\}_{n=1}^{\infty} \) if there is a subsequence of \( \{a_n\}_{n=1}^{\infty} \) which converges to \( \lambda \).

4. Let \( \lambda \) be a subsequential limit of a bounded sequence \( \{a_n\}_{n=1}^{\infty} \). Prove that \( \liminf_{n \to \infty} a_n \leq \lambda \leq \limsup_{n \to \infty} a_n \).

5. Suppose that \( \{a_n\}_{n=1}^{\infty} \) is a bounded sequence. Show that the following are equivalent:
   (a) \( M = \limsup_{n \to \infty} a_n \).
   (b) \( M \) is the largest subsequential limit of \( \{a_n\}_{n=1}^{\infty} \).

6. Suppose that \( \{a_n\}_{n=1}^{\infty} \) is a bounded sequence. Show that the following are equivalent:
   (a) \( m = \liminf_{n \to \infty} a_n \).
   (b) \( m \) is the smallest subsequential limit of \( \{a_n\}_{n=1}^{\infty} \).

7. (Ratio Test) (i) Let \( \{a_n\}_{n=1}^{\infty} \) be a sequence of nonzero real numbers, and let
   \[
   a := \liminf_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \quad \text{and} \quad A := \limsup_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|.
   \]
   Prove the following:
   (a) If \( A < 1 \), then the series \( \sum_{n=1}^{\infty} a_n \) converges absolutely.
   (b) If \( a > 1 \), then the series \( \sum_{n=1}^{\infty} a_n \) diverges.
   (ii) Give examples to show that no conclusion can be drawn if \( a \leq 1 \leq A \).

8. (Root Test) (i) Let \( \{a_n\}_{n=1}^{\infty} \) be a sequence of real numbers. Prove the following statements:
   (a) If \( \limsup_{n \to \infty} |a_n|^{1/n} = A < 1 \), then the series \( \sum_{n=1}^{\infty} a_n \) converges absolutely.
   (b) If \( \limsup_{n \to \infty} |a_n|^{1/n} = A > 1 \), or if \( \limsup_{n \to \infty} |a_n|^{1/n} = +\infty \), then the series \( \sum_{n=1}^{\infty} a_n \) diverges.
   (ii) Give examples to show that no conclusion can be drawn if \( \limsup_{n \to \infty} |a_n|^{1/n} = 1 \).