Assignment 6

1. Suppose $\{d_n\}_{n=1}^{\infty}$ is sequence of positive numbers such that the infinite series $\sum_{n=1}^{\infty} d_n$ is divergent. Decide on the convergence/divergence of each of the following series; justify your assertions.

$$\text{(i)}\quad \sum_{n=1}^{\infty} \frac{d_n}{1+n^2 d_n} \qquad \text{(ii)}\quad \sum_{n=1}^{\infty} \frac{d_n}{1+d_n} \qquad \text{(iii)}\quad \sum_{n=1}^{\infty} \frac{d_n}{1+d_n^2}$$

2. Let A be a fixed positive number and consider the problem of finding the square root of A. This problem is tantamount to solving the equation $f(x) = 0$, where $f(x) := x^2 - A$. The *Newton-Raphson method* provides a certain algorithm to solve the equation; specifically, one considers the sequence $\{x_n\}$ defined recursively:

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n \geq 1,$$

where x_1 is a certain specified number.

(i) Verify that for $f(x) = x^2 - A$, the recursion above reduces to the following:

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{A}{x_n}\right], \qquad n \geq 1.$$

(ii) Let the initial value x_1 be chosen such that $x_1 > 0$ and $x_1^2 > A$. Show that the resulting sequence $\{x_n\}$ is nonincreasing and bounded below.

(iii) Show that $\lim_{n\to\infty} x_n = \sqrt{A}$.

3. Suppose $\{s_n\}$ is a bounded sequence of real numbers. Let

$$\sigma_n := \frac{s_1 + \cdots + s_n}{n}, \qquad n \in \mathbf{N}.$$

(i) Show that $\{\sigma_n\}$ is a bounded sequence.

(ii) Prove that

$$\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} s_n\,.$$

(iii) Deduce the following result from (ii): If $\lim_{n\to\infty} s_n = L$, then $\lim_{n\to\infty} \sigma_n = L\,.$

(iv) Find a bounded, divergent sequence $\{s_n\}$ such that $\{\sigma_n\}$ is convergent. (This will demonstrate that the converse of (iii) is false in general.)

4. Show that a nonempty subset of $\mathbf{R}$ is open if and only if it is the union of open intervals.

Suppose that A is a nonempty subset of the real line. A real number x is said to be an *interior point* of A if there is an open interval I_x such that $x \in I_x \subseteq A$. The set of all interior points of A is called the *interior* of A; it is denoted by A°.

5. Find the interior of each of the following sets:
(i) The interval $(0, 1)$
(ii) The interval $[0, 1]$
(iii) The interval $(-\infty, 3]$
(iv) The set of rational numbers
(v) The set of irrational numbers
(vi) The set of integers

6. Suppose that A is a nonempty subset of the real line. Prove the following statements:
(i) $A^\circ \subseteq A$.
(ii) A° is an open set.
(iii) A° is the largest open set contained in A, that is, if G is any open set contained in A, then $G \subseteq A^\circ$.
(iv) A is open if and only if $A = A^\circ$.

7. Suppose that A and B are nonempty subsets of $\mathbf{R}$.
(i) Show that $(A \cap B)^\circ = A^\circ \cap B^\circ$.
(ii) Show that $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$.
(iii) Give an example to show that $A^\circ \cup B^\circ$ need not equal $(A \cup B)^\circ$.

Suppose that A is a nonempty subset of the real line. We say that a real number x belongs to the *closure* of A if $(x - \epsilon, x + \epsilon) \cap A$ is nonempty for *every* positive number ϵ. The collection of all such points x is called the closure of A, and it is denoted by $\overline{A}$.

8. Find the closure of each of the subsets given in Question 5.

9. Suppose that A is a nonempty subset of the real line. Prove the following statements:
(i) $A \subseteq \overline{A}$.
(ii) $\overline{A}$ is a closed set.
(iii) $\overline{A}$ is the smallest closed set containing A, that is, if F is any closed set containing A, then $\overline{A} \subseteq F$.
(iv) A is closed if and only if $A = \overline{A}$.

10. Suppose that A is a nonempty subset of the real line. Prove that the following statements are equivalent:
(a) $x \in \overline{A}$.
(b) There is a sequence in A which converges to x.

11. Suppose that A and B are nonempty subsets of $\mathbf{R}$.
(i) Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
(ii) Show that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.
(iii) Give an example to show that $\overline{A \cap B}$ need not equal $\overline{A} \cap \overline{B}$.