1. Suppose \( \{d_n\}_{n=1}^{\infty} \) is sequence of positive numbers such that the infinite series \( \sum_{n=1}^{\infty} d_n \) is divergent. Decide on the convergence/divergence of each of the following series; justify your assertions.

\[
\begin{align*}
(i) & \quad \sum_{n=1}^{\infty} \frac{d_n}{1 + n^2 d_n} & (ii) & \quad \sum_{n=1}^{\infty} \frac{d_n}{1 + d_n} & (iii) & \quad \sum_{n=1}^{\infty} \frac{d_n}{1 + d_n^2}
\end{align*}
\]

2. Let \( A \) be a fixed positive number and consider the problem of finding the square root of \( A \). This problem is tantamount to solving the equation \( f(x) = 0 \), where \( f(x) := x^2 - A \). The Newton-Raphson method provides a certain algorithm to solve the equation; specifically, one considers the sequence \( \{x_n\} \) defined recursively:

\[
x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 1,
\]

where \( x_1 \) is a certain specified number.

(i) Verify that for \( f(x) = x^2 - A \), the recursion above reduces to the following:

\[
x_{n+1} = \frac{1}{2} \left[ x_n + \frac{A}{x_n} \right], \quad n \geq 1.
\]

(ii) Let the initial value \( x_1 \) be chosen such that \( x_1 > 0 \) and \( x_1^2 > A \). Show that the resulting sequence \( \{x_n\} \) is nonincreasing and bounded below.

(iii) Show that \( \lim_{n \to \infty} x_n = \sqrt{A} \).

3. Suppose \( \{s_n\} \) is a bounded sequence of real numbers. Let

\[
\sigma_n := \frac{s_1 + \cdots + s_n}{n}, \quad n \in \mathbb{N}.
\]

(i) Show that \( \{\sigma_n\} \) is a bounded sequence.

(ii) Prove that

\[
\liminf_{n \to \infty} s_n \leq \liminf_{n \to \infty} \sigma_n \leq \limsup_{n \to \infty} \sigma_n \leq \limsup_{n \to \infty} s_n.
\]

(iii) Deduce the following result from (ii): If \( \lim_{n \to \infty} s_n = L \), then \( \lim_{n \to \infty} \sigma_n = L \).

(iv) Find a bounded, divergent sequence \( \{s_n\} \) such that \( \{\sigma_n\} \) is convergent. (This will demonstrate that the converse of (iii) is false in general.)

4. Show that a nonempty subset of \( \mathbb{R} \) is open if and only if it is the union of open intervals.

Suppose that \( A \) is a nonempty subset of the real line. A real number \( x \) is said to be an interior point of \( A \) if there is an open interval \( I_x \) such that \( x \in I_x \subseteq A \). The set of all interior points of \( A \) is called the interior of \( A \); it is denoted by \( A^\circ \).
5. Find the interior of each of the following sets:
   (i) The interval \((0, 1)\)
   (ii) The interval \([0, 1]\)
   (iii) The interval \((-\infty, 3]\)
   (iv) The set of rational numbers
   (v) The set of irrational numbers
   (vi) The set of integers

6. Suppose that \(A\) is a nonempty subset of the real line. Prove the following statements:
   (i) \(A^o \subseteq A\).
   (ii) \(A^o\) is an open set.
   (iii) \(A^o\) is the largest open set contained in \(A\), that is, if \(G\) is any open set contained in \(A\), then \(G \subseteq A^o\).
   (iv) \(A\) is open if and only if \(A = A^o\).

7. Suppose that \(A\) and \(B\) are nonempty subsets of \(\mathbb{R}\).
   (i) Show that \((A \cap B)^o = A^o \cap B^o\).
   (ii) Show that \(A^o \cup B^o \subseteq (A \cup B)^o\).
   (iii) Give an example to show that \(A^o \cup B^o\) need not equal \((A \cup B)^o\).

Suppose that \(A\) is a nonempty subset of the real line. We say that a real number \(x\) belongs to the closure of \(A\) if \((x - \epsilon, x + \epsilon) \cap A\) is nonempty for every positive number \(\epsilon\). The collection of all such points \(x\) is called the closure of \(A\), and it is denoted by \(\overline{A}\).

8. Find the closure of each of the subsets given in Question 5.

9. Suppose that \(A\) is a nonempty subset of the real line. Prove the following statements:
   (i) \(A \subseteq \overline{A}\).
   (ii) \(\overline{A}\) is a closed set.
   (iii) \(\overline{A}\) is the smallest closed set containing \(A\), that is, if \(F\) is any closed set containing \(A\), then \(\overline{A} \subseteq F\).
   (iv) \(A\) is closed if and only if \(A = \overline{A}\).

10. Suppose that \(A\) is a nonempty subset of the real line. Prove that the following statements are equivalent:
    (a) \(x \in \overline{A}\).
    (b) There is a sequence in \(A\) which converges to \(x\).

11. Suppose that \(A\) and \(B\) are nonempty subsets of \(\mathbb{R}\).
    (i) Show that \(\overline{A \cup B} = \overline{A} \cup \overline{B}\).
    (ii) Show that \(\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}\).
    (iii) Give an example to show that \(\overline{A \cap B}\) need not equal \(\overline{A} \cap \overline{B}\).