Assignment 7

Definition 1. Suppose that f is a function defined in an interval (c,d), $d > c$. Let L be a real number. We say that $\lim_{x\to c^+} f(x) = L$ if the following holds: for every sequence $\{x_n\}_{n=1}^{\infty}$ such that $c < x_n < d$ for every n, and $\lim_{n\to\infty} x_n = c$, we have $\lim_{n\to\infty} f(x_n) = L$.

1. Suppose that f is a function defined in an interval (c,d), $d > c$. Let L be a real number. Show that the following statements are equivalent:
(a) $\lim_{x\to c^+} f(x) = L$.
(b) Given $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < x - c < \delta$.

2. Suppose that f is a function defined in an interval (b,c), $b < c$, and let L be a real number.
(i) Formulate a definition (along the lines of Definition 1 above) for $\lim_{x\to c^-} f(x) = L$.
(ii) State and prove an equivalent formulation of (i), as in Exercise 1 above.

Definition 2. Suppose that f is a function defined in an interval (c,d), $d > c$. We say that $\lim_{x\to c^+} f(x) = +\infty$ if the following holds: for every sequence $\{x_n\}_{n=1}^{\infty}$ such that $c < x_n < d$ for every n, and $\lim_{n\to\infty} x_n = c$, we have $\lim_{n\to\infty} f(x_n) = +\infty$.

3. Suppose that f is a function defined in an interval (c,d), $d > c$. Show that the following statements are equivalent:
(a) $\lim_{x\to c^+} f(x) = +\infty$.
(b) Given $T > 0$, there is a $\delta > 0$ such that $f(x) > T$ whenever $0 < x - c < \delta$.

4. Formulate a definition for $\lim_{x\to c^+} = -\infty$, and prove an equivalent formulation as in the previous exercise.

5. Carry out the entire programme above for $\lim_{x\to c^-} f(x) = +\infty$ and $\lim_{x\to c^-} f(x) = -\infty$.

Definition 3. Suppose that f is defined on some interval (A,∞), where A is a fixed real number. Let L be a fixed real number. We say that $\lim_{x\to\infty} f(x) = L$ if the following holds: for every sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n > A$ for every n, and $\lim_{n\to\infty} x_n = +\infty$, we have $\lim_{n\to\infty} f(x_n) = L$.

6. Suppose that f is defined on some interval (A,∞), where A is a fixed real number. Let L be a fixed real number. Prove that the following are equivalent:
(a) $\lim_{x\to\infty} f(x) = L$.
(b) Given $\epsilon > 0$, there is an $R > 0$ such that $|f(x) - L| < \epsilon$ for every $x > R$.

7. Formulate appropriate definitions and prove the corresponding equivalent statements as above, for each of the following: $\lim_{x\to\infty} f(x) = \pm\infty$, $\lim_{x\to-\infty} f(x) = L$ (L a real number), and $\lim_{x\to-\infty} f(x) = \pm\infty$.

8. Suppose that f is defined on a domain D, and that $x_0 \in D$. Prove that the following are equivalent:

(a) If $\{x_n\}_{n=1}^{\infty}$ is a sequence in D such that $\lim_{n\to\infty} x_n = x_0$, then $\lim_{n\to\infty} f(x_n) = f(x_0)$.

(b) If $\{x_n\}_{n=1}^{\infty}$ is a sequence in $D \setminus \{x_0\}$ such that $\lim_{n\to\infty} x_n = x_0$, then $\lim_{n\to\infty} f(x_n) = f(x_0)$.

9. Finish the following proof that was mentioned in lecture: Let f be a function defined on a domain D. Then f is continuous at a point x_0 in D if and only if, given $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $x \in D$ and $|x - x_0| < \delta$.

10. Suppose that f is defined in an open interval containing a point c. Prove that f is continuous at c if and only if $\lim_{x\to c} f(x) = f(c)$.

Definition 4. Suppose that X and Y are nonempty sets, and that $f : X \to Y$ is a function. Given $B \subseteq Y$, we define the *inverse image of B under f* as follows:

$$I_f(B) := \{x \in X : f(x) \in B\}.$$

11. Suppose that X and Y are nonempty sets, and that $f : X \to Y$ is a function. Prove the following statements:

(i) If B_1 and B_2 are subsets of Y, then $I_f(B_1 \cup B_2) = I_f(B_1) \cup I_f(B_2)$.

(ii) If B_1 and B_2 are subsets of Y, then $I_f(B_1 \cap B_2) = I_f(B_1) \cap I_f(B_2)$.

(iii) If B is a subset of Y, then $I_f(B^c) = (I_f(B))^c$, where the superscript denotes complementation.

11. Let $f : \mathbf{R} \to \mathbf{R}$ be a function. Prove that the following statements are equivalent:

(a) f is continuous on $\mathbf{R}$ (recall that this means that f is continuous at every real number).

(b) If G is an open subset of $\mathbf{R}$, then $I_f(G)$ is an open subset as well.

(c) If H is a closed subset of $\mathbf{R}$, then $I_f(H)$ is a closed subset as well.