

MATH 409-08a, Quiz 12

SOLUTIONS

1. We have the inequalities

$$m \leq f(x) \leq M, \quad a \leq x \leq b.$$

A theorem discussed in lecture allows us to integrate each term above, and preserve the same inequalities; thus

$$\int_a^b m \, dx \leq \int_a^b f(x) \, dx \leq \int_a^b M \, dx,$$

which gives

$$m(b-a) \leq \int_a^b f(x) \, dx \leq M(b-a).$$

Now the relations above can be recast as follows:

$$m \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq M.$$

Recall from the Extreme Value Theorem that there exist points α_* and α^* in $[a, b]$ such that $m = f(\alpha_*)$ and $M = f(\alpha^*)$. Thus we have

$$f(\alpha_*) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq f(\alpha^*),$$

whence the Intermediate Value Theorem provides a $c \in [a, b]$ for which

$$f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx.$$

2. Consider

$$F(x) = \int_0^x e^{\sqrt{t^2+1}} \, dt, \quad x \in \mathbf{R}.$$

Assume that $h > 0$. Then

$$F(2+h) - F(2) = \int_0^{2+h} e^{\sqrt{t^2+1}} \, dt - \int_0^2 e^{\sqrt{t^2+1}} \, dt = \int_2^{2+h} e^{\sqrt{t^2+1}} \, dt. \quad (1)$$

On the other hand, if $h < 0$, then

$$F(2) - F(2+h) = \int_0^2 e^{\sqrt{t^2+1}} \, dt - \int_0^{2+h} e^{\sqrt{t^2+1}} \, dt = \int_{2+h}^2 e^{\sqrt{t^2+1}} \, dt = - \int_2^{2+h} e^{\sqrt{t^2+1}} \, dt. \quad (2)$$

Combining (1) and (2), we find that, for every $h \neq 0$,

$$\frac{1}{h} \int_2^{2+h} e^{\sqrt{t^2+1}} \, dt = \frac{F(2+h) - F(2)}{h}. \quad (3)$$

The function $f(t) := e^{\sqrt{t^2+1}}$ is continuous on \mathbf{R} , so the Fundamental Theorem of Calculus asserts that $F'(2) = f(2)$. Consequently, we find from (3) (and the definition of the derivative) that

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_2^{2+h} e^{\sqrt{t^2+1}} \, dt = \lim_{h \rightarrow 0} \frac{F(2+h) - F(2)}{h} = F'(2) = f(2) = e^{\sqrt{5}}.$$