

MATH 409-08a, Quiz 5

SOLUTIONS

1. (i) All superior limits in question exist, because each of the sequences $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and hence $\{a_n + b_n\}_{n=1}^{\infty}$, is bounded. Let

$$M_n(a) := \sup\{a_k : k \geq n\}, \quad M_n(b) := \sup\{b_k : k \geq n\}, \quad \text{and} \quad M_n(a+b) := \sup\{a_k + b_k : k \geq n\}.$$

If $k \geq n$, then $a_k + b_k \leq M_n(a) + M_n(b)$, so $M_n(a) + M_n(b)$ is an upper bound for the collection of numbers $\{a_k + b_k : k \geq n\}$. Therefore the least upper bound for this collection, namely $M_n(a+b)$ satisfies the estimate

$$M_n(a+b) \leq M_n(a) + M_n(b), \quad n \in \mathbf{N}.$$

Letting n approach infinity on both sides of the inequality above yields

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n.$$

(ii) Let $a_n := (-1)^n$ and $b_n := (-1)^{n-1}$, $n \in \mathbf{N}$. Then $a_n + b_n = 0$ for every n , so $\limsup_{n \rightarrow \infty} (a_n + b_n) = 0$. On the other hand, $\limsup_{n \rightarrow \infty} a_n = 1 = \limsup_{n \rightarrow \infty} b_n$.

2. We begin by showing that (a) implies (b). Assume that $m = \liminf_{n \rightarrow \infty} a_n$, and let $\epsilon > 0$ be given. Let $m_n = \inf\{a_k : k \geq n\}$, and recall that $m = \sup\{m_n : n \in \mathbf{N}\}$. As $m - \epsilon < m$, there is an $m_p > m - \epsilon$ for some positive integer p . As $m_p \leq a_k$ for every $k \geq p$, we find that $a_k > m - \epsilon$ for every such k . In particular, the set of k 's for which this inequality does not hold is at most finite. Now consider the number $m + \epsilon$. If the required result weren't to hold, then we must have $a_n < m + \epsilon$ for finitely many n . So there exists a positive integer r such that $a_n \geq m + \epsilon$ for every $n \geq r$. Therefore $m_r = \inf\{a_k : k \geq r\} \geq m + \epsilon$, whence we obtain

$$m = \sup\{m_n : n \in \mathbf{N}\} \geq m_r > m + \epsilon,$$

contradicting the fact that ϵ is positive.

We now prove that (b) implies (a). Assume that m satisfies the pair of conditions given in (b), and let $\{m_n\}_{n=1}^{\infty}$ be the sequence defined above. We must show that (i) $m_n \leq m$ for every n , and (ii) given $m' < m$, there is some $m_j > m'$. If (i) fails, then there is some $m_p > m$. Choose an $\epsilon > 0$ such that $m_p > m + \epsilon > m$. If $k \geq p$, then $a_k \geq m_p > m + \epsilon$, but this means that the inequality $a_n < m + \epsilon$ cannot hold for infinitely many values of n , contradicting the second assumption in (b). Thus (i) is proven. Turning to (ii), let $m' < m$ be given, and choose an $\epsilon > 0$ such that $m' < m - \epsilon$. By assumption, $a_n > m - \epsilon$ for all but at most finitely many values of n , so there is a positive integer j such that $a_n > m - \epsilon$ for every $n \geq j$. Therefore $m_j = \inf\{a_k : k \geq j\} \geq m - \epsilon > m'$. This proves (ii).