MATH 409-08a, Quiz 9

SOLUTIONS

(i) We must show the following: given $\epsilon > 0$, there is a $\delta > 0$ such that $|f(s) - f(t)| < \epsilon$ whenever $s, t \geq 0$ and $|s - t| < \delta$.

Let $\epsilon > 0$ be given. As f is given to be uniformly continuous on $[a, \infty)$, there is a $\delta_1 > 0$ such that

$$|f(u) - f(v)| < \epsilon/2 \qquad \text{whenever} \quad u, v \geq a, \quad |u - v| < \delta_1. \tag{1}$$

As f is continuous on the compact set $[0, a]$, it must be uniformly continuous there. So there is a $\delta_2 > 0$ such that

$$|f(u) - f(v)| < \epsilon/2 \qquad \text{whenever} \quad 0 \leq u, v \leq a, \quad |u - v| < \delta_2. \tag{2}$$

Choose $\delta := \min\{\delta_1, \delta_2\}$, and let $|s - t| < \delta$. Let $s, t \in [a, \infty)$. As $\delta \leq \delta_1$, we find from (1) that $|f(s) - f(t)| < \epsilon/2 < \epsilon$. Likewise, (2) provides the estimate $|f(s) - f(t)| < \epsilon$ if $s, t \in [0, a]$. Suppose now that $s < a < t$. The relation $|s - t| < \delta$ implies that each of the numbers $|s - a|$ and $|t - a|$ is also smaller than δ. In particular, $|s - a| < \delta_1$ and $|t - a| < \delta_2$. Therefore (1) and (2) ensure that $|f(s) - f(a)| < \epsilon/2$ and $|f(a) - f(t)| < \epsilon/2$. Consequently, the triangle inequality implies that

$$|f(s) - f(t)| = |f(s) - f(a) + f(a) - f(t)| \leq |f(s) - f(a)| + |f(a) - f(t)| < \epsilon.$$

(ii) As the function $[0, \infty) \ni x \mapsto \sqrt{x}$ is continuous on $[0, \infty)$, it suffices to prove, thanks to part (i) above, that this function is uniformly continuous on $[1, \infty)$. To that end, let $s, t \geq 1$ and observe that

$$|\sqrt{s} - \sqrt{t}| = \frac{|s - t|}{\sqrt{s} + \sqrt{t}} \leq \frac{|s - t|}{2}, \tag{3}$$

the final inequality stemming from the fact that $\sqrt{s} + \sqrt{t} \geq 2$ (because $s, t \geq 1$). We find from (3) that the function $x \mapsto \sqrt{x}$ is Lipschitz, hence uniformly continuous, on $[1, \infty)$.