Exercise Set 4

1. Suppose that $V$ is a vector space. Show that the discrete metric on $V$ cannot be induced by any norm on $V$.

The following question is taken from a 1993 examination given in the University of Toronto. The examinees were students of electrical engineering, taking a course in Data Communication.

2. For a particular signal processing problem, Engineer A defines a function $d(x, y)$ for all $x, y,$ and $z$ belonging to a set $S$ with the following properties:
   (i) $d(x, y) = 0$ if and only if $x = y$.
   (ii) $d(x, y) = d(y, x)$.
   (iii) $d(x, z) \leq d(x, y) + d(y, z)$.
   Engineer A claims that $d(x, y)$ is a metric.
   (a) Engineer B points out that $d$ should satisfy one more condition before it can be called a metric. Is he right? If so, what is the condition?
   (b) Engineer A asserts that his function is still a metric, nevertheless. Is he right?

3. Suppose that $M$ is a nonempty set and $D : M \times M \to \mathbb{R}$ is a function satisfying the following conditions:
   (i) $D(a, a) = 0$ for every $a \in M$.
   (ii) $D(a, b) \neq 0$ if $a \neq b$.
   (iii) $D(a, b) + D(b, c) \geq D(c, a)$ for every $a, b, c \in M$.
   Show that $D$ is a metric on $M$.

4. Suppose that $(X, d)$ is a metric space. Show that the function $D(x, y) := \frac{d(x, y)}{1 + d(x, y)}$, $x, y \in X$, is a metric on $X$. 