Exercise Set 6

**Definition.** Suppose $V$ is a linear space. A function $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ is said to be an inner product on $V$ if the following conditions are satisfied:

1. **(IP1)** $\langle x, y \rangle = \langle y, x \rangle$ for every $x, y \in V$.
2. **(IP2)** $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in V$.
3. **(IP3)** $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ for all $x, y \in V$ and every $\alpha \in \mathbb{R}$.
4. **(IP4)** $\langle x, x \rangle \geq 0$ for every $x \in V$ and $\langle x, x \rangle = 0$ if and only if $x = \mathbf{0}$, where $\mathbf{0}$ denotes the zero vector in $V$.

The pair $(V, \langle \cdot, \cdot \rangle)$ is called an inner-product space.

1. Suppose $(V, \langle \cdot, \cdot \rangle)$ is an inner product space. Prove the following statements:
   (i) Show that $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$ for all $x, y, z \in V$.
   (ii) Show that $\langle x, \alpha y \rangle = \alpha \langle x, y \rangle$ for all $x, y \in V$ and every $\alpha \in \mathbb{R}$.
   (iii) Show that $\langle x, \mathbf{0} \rangle = 0$ for every $x \in V$.

2. Suppose $(V, \langle \cdot, \cdot \rangle)$ is an inner product space. Prove the Bunyakowski-Cauchy-Schwarz inequality:

$$|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \sqrt{\langle y, y \rangle}, \quad \forall x, y \in V.$$  

**Suggestion:** First dispose of the case when $x$ or $y$ is the zero vector. If neither $x$ nor $y$ is zero, consider the polynomial $p(t) := \langle x + ty, x + ty \rangle$, $t \in \mathbb{R}$, and exploit the fact that $p(t) \geq 0$ for every $t$ (because of Condition (IP4) of an inner product).

3. Suppose $(V, \langle \cdot, \cdot \rangle)$ is an inner product space. Define

$$\|x\| := \sqrt{\langle x, x \rangle}, \quad x \in V.$$ 

Prove that $\| \cdot \|$ is a norm on $V$. This is called the norm induced by the inner product.

4. Suppose $(V, \langle \cdot, \cdot \rangle)$ is an inner product space, and let $\| \cdot \|$ denote the norm induced by the inner product. Establish the Parallelogram Identity:

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2), \quad \forall x, y \in V.$$ 

5. (i) Show that $\ell^2_n$ is an inner-product space (more specifically, find an inner product on $\mathbb{R}^n$ which induces the Euclidean norm $\| \cdot \|_2$).
   (ii) Use Question 5 to show that $\ell^p_n$ is not an inner-product space for $p \in [1, 2) \cup (2, \infty]$ (that is, the $p$-norm $\| \cdot \|_p$ on $\mathbb{R}^n$ cannot be induced by an inner product for $p \neq 2$).

6. (i) Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences in $\ell^2$. Show that the sequence $\{c_n\}$, defined by $c_n := a_n b_n$, $n \in \mathbb{N}$, belongs to $\ell^1$.
   (ii) Show that the function $\langle \cdot, \cdot \rangle : \ell^2 \times \ell^2 \rightarrow \mathbb{R}$ given by

$$\langle x, y \rangle := \sum_{n=1}^{\infty} x_n y_n, \quad x = \{x_n\}, \ y = \{y_n\} \in \ell^2,$$

is an inner product on $\ell^2$.
   (iii) Verify that the norm induced by the inner product above is the usual norm on $\ell^2$. (Thus $\ell^2$ is an inner-product space which is complete with respect to the norm induced by the inner product; such spaces are called Hilbert Spaces.)
   (iv) Show that the $\ell^p$-norm is not induced by an inner product for any $p \neq 2$. 

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