Exercise Set 9

1. Suppose $X$ and $Y$ are nonempty sets and $f : X \to Y$.
   (i) Show that $f(f^{-1}(B)) \subseteq B$ for every $B \subseteq Y$.
   (ii) Show that $A \subseteq f^{-1}(f(A))$ for every $A \subseteq X$.

2. Suppose $(X, \rho)$ and $(Y, \rho')$ are metric spaces and $f : X \to Y$. Prove that $f$ is continuous on $X$ if and only if $f(A) \subseteq f(A)$ for every $A \subseteq X$.

3. Suppose that $(M, \rho)$ and $(M', \rho')$ are metric spaces, and let $f$ and $g$ be functions from $M$ to $M'$. Assume that both $f$ and $g$ are continuous on $M$, and that $f(x) = g(x)$ for every $x \in D$, where $D$ is a dense subset of $M$. Show that $f(x) = g(x)$ for every $x \in M$.

4. Suppose that $(X, \rho)$ is a metric space, and let $T$ denote the set $\{0, 1\}$ equipped with the discrete metric. Prove that the following statements are equivalent:
   (i) $X$ is connected.
   (ii) If $f : X \to T$ is continuous, then $f$ is constant.

5. Give a direct proof of Theorem 3.8.9 (that is, show that given $\epsilon > 0$, there is a $\delta > 0$ such that $\rho'(f(s), f(t)) < \epsilon$ whenever $\rho(s, t) < \delta$).

6. Suppose $(X, d)$ is a metric space.
   (i) Let $B$ be a nonempty subset of $X$. Define $f : X \to \mathbb{R}$ as follows:
      \[ f(x) := \inf \{ d(x, b) : b \in B \} , \quad x \in X . \]
   (a) Show that $f(x) = 0$ if and only if $x$ belongs to the closure of $B$.
   (b) Prove that $|f(x) - f(y)| \leq d(x, y)$, $x, y \in X$.
   (c) Deduce that $f$ is uniformly continuous on $X$ (assuming that $\mathbb{R}$ is equipped with the usual metric).
   (ii) Show that $(X, d)$ is completely regular, to wit: given $x_0 \in X$ and a closed set $B$ which does not contain $x_0$, there is a continuous function $g : X \to \mathbb{R}$ (where $\mathbb{R}$ is equipped with the usual metric) such that $g(x_0) = 1$ and $g(x) = 0$ for every $x \in B$.
   (iii) Deduce that $(X, d)$ is regular, that is, given $x_0 \in X$ and a closed set $B$ which does not contain $x_0$, there exist $d$-open sets $U$ and $V$ such that $x_0 \in U$, $B \subseteq V$, and $U \cap V = \emptyset$.

7. Suppose that $(X, d)$ is a metric space and let the real line $\mathbb{R}$ be equipped with the usual metric. Let $f : X \to \mathbb{R}$ be continuous on $X$. Show that the zero-set of $f$, namely,$\[ Z(f) := \{ p \in X : f(p) = 0 \} \]
   is a closed subset of $X$. 
8. Suppose that \((X, d)\) is a metric space. Given a closed subset \(S\) of \(X\), define

\[
f_S(x) := \inf\{d(x, s) : s \in S\}
\]

(i) Recall from Example 6 above that \(f_S\) is continuous on \(X\) and that \(f_S(x) = 0\) if and only if \(x \in S\).

(ii) Use this function to show that the converse of Example 1 (above) holds, that is, every closed subset of \(X\) is a zero set of some continuous function from \(X\) to \(\mathbb{R}\).

(iii) Let \(A\) and \(B\) be a pair of disjoint nonempty closed subsets of \(X\). Define

\[
G(x) := \frac{f_A(x)}{f_A(x) + f_B(x)}, \quad x \in X.
\]

(a) Show that \(G\) is (well defined and) continuous on \(X\).

(b) Verify that the range of \(G\) is contained in \([0, 1]\).

(c) Show that \(G(x) = 0\) if and only if \(x \in A\) and \(G(x) = 1\) if and only if \(x \in B\).

(The result developed in (iii) is the metric-space version of the so-called Urysohn’s Lemma, a celebrated theorem in general topology.)

(iv) Prove that \((X, d)\) is normal, that is, for every pair of disjoint nonempty closed subsets \(A\) and \(B\) of \(X\), there exist disjoint open sets \(U\) and \(V\) in \(X\) such that \(A \subseteq U\) and \(B \subseteq V\).

**Definition.** Suppose that \((M, \rho)\) and \((M', \rho')\) are metric spaces. A function \(H : M \to M'\) is called a homeomorphism if it satisfies each of the following conditions: (i) \(H\) is a bijection, (ii) \(H\) is continuous on \(M\), and (iii) \(H(U)\) is \((\rho')\)-open in \(M'\) whenever \(U\) us \((\rho)\)-open in \(M\). Two metric spaces are said to be homeomorphic if there exists a homeomorphism from one onto the other.

9. Suppose that \((M, \rho)\) and \((M', \rho')\) are metric spaces. Let \(H : M \to M'\) be a bijection, and let \(H^{-1} : M' \to M\) denote its inverse. Show that the following conditions are equivalent:

(i) \(H\) is a homeomorphism.

(ii) If \(C\) is any closed subset of \(M\), then \(H(C)\) is a closed subset of \(M'\).

(iii) \(H\) and \(H^{-1}\) are continuous.

10. (i) Suppose that \((M, \rho)\) is a metric space, and that \(A\) is a (nonempty) subset of \(M\). Prove that, if \((A, \rho)\) is compact, then \(A\) is a closed subset of \(M\).

(ii) Suppose that \((M, \rho)\) is a compact metric space, and that \(A\) is a (nonempty) closed subset of \(M\). Prove that \((A, \rho)\) is compact.

11. Suppose that \((M, \rho)\) and \((M', \rho')\) are metric spaces, and that \((M, \rho)\) is compact. Let \(F : M \to M'\) be a bijection. Prove that \(F\) is a homeomorphism if and only if it is continuous on \(M\).

12. Assume that the real line \(\mathbb{R}\) is equipped with the usual metric. Let \(f\) be a function from \(\mathbb{R}\) into itself. Assume that \(f\) is continuous on \(\mathbb{R}\) and \(\lim_{x \to \pm \infty} f(x) = 0\). Prove that \(f\) is uniformly continuous on \(\mathbb{R}\).

13. Suppose \(A\) is a nonempty bounded subset of the real line (equipped with the usual metric). Prove that \(A\) is compact if and only if every continuous function from \(A\) to \(\mathbb{R}\) is uniformly continuous (on \(A\)).
Definition. Suppose \( f : \mathbb{R} \to \mathbb{R} \) is a bounded function (that is, there is some \( \Delta > 0 \) such that \( |f(x)| \leq \Delta \) for every real number \( x \)). Let \( J \) be any bounded open interval in \( \mathbb{R} \). The oscillation of \( f \) over \( J \) is defined as follows:

\[
\omega[f; J] := \sup_{x \in J} f(x) - \inf_{x \in J} f(x).
\]

Given a real number \( a \), we define the oscillation of \( f \) at \( a \) as follows:

\[
\omega[f; a] := \inf\{\omega[f; J] : J \text{ is a bounded open interval containing } a\}.
\]

14. Suppose that \( f \) and \( J \) are as above. Show that

\[
\omega[f; J] = \sup\{|f(u) - f(v)| : u, v \in J\}.
\]

15. Suppose that \( \mathbb{R} \) is equipped with the usual metric and that \( f : \mathbb{R} \to \mathbb{R} \) is a bounded function. Let \( a \) be a fixed real number. Prove that \( f \) is continuous at \( a \) if and only if \( \omega[f; a] = 0 \).

16. Suppose that \( \mathbb{R} \) is equipped with the usual metric and that \( f : \mathbb{R} \to \mathbb{R} \) is a bounded function. Let \( \alpha \) be a fixed positive number. Show that the set

\[
S := \{a \in \mathbb{R} : \omega[f; a] \geq \alpha\}
\]

is a closed subset of \( \mathbb{R} \).

Definition. A subset of a metric space is called an \( F_\sigma \)-set if it is the countable union of closed sets.

17. Suppose that \( \mathbb{R} \) is equipped with the usual metric and that \( f : \mathbb{R} \to \mathbb{R} \) is a bounded function. Define

\[
\mathcal{D}(f) := \{a \in \mathbb{R} : f \text{ is discontinuous at } a\}.
\]

Prove that \( \mathcal{D} \) is an \( F_\sigma \)-set.

18. Suppose that \( (V, \|\cdot\|_V) \) and \( (W, \|\cdot\|_W) \) are normed linear spaces, and let \( T : V \to W \) be a linear transformation (recall that this means \( T(x + y) = T(x) + T(y) \) and \( T(\alpha x) = \alpha T(x) \) for every \( x, y \in V \) and every real number \( \alpha \)). Let \( 0_V \) and \( 0_W \) denote the zero elements in \( V \) and \( W \), respectively. Recall that \( T(0_V) = 0_W \). Prove that the following statements are equivalent:

(i) \( T \) is continuous on \( V \) (i.e., with respect to the metrics induced by the norms).

(ii) \( T \) is continuous at \( 0_V \).

(iii) There is a positive constant \( K \) such that \( \|T(x)\|_W \leq K \|x\|_V \) for every \( x \in V \).

(Show that (i) and (ii) are equivalent, and that (ii) and (iii) are equivalent.)

Definition. Suppose \( f \) is a function from \( \mathbb{R} \) to \( \mathbb{R} \). We say that \( f \) has the Intermediate Value Property if the following holds: whenever \( f(x) < T < f(y) \), there is some \( z \) between \( x \) and \( y \) such that \( f(z) = T \).

19. Suppose \( f : \mathbb{R} \to \mathbb{R} \) is a given function. Prove that \( f \) is continuous on \( \mathbb{R} \) (with respect to the usual metric) if and only if the following pair of conditions holds:

(a) \( f \) has the Intermediate Value Property.

(b) \( f^{-1}([r]) \) is a closed subset of \( \mathbb{R} \) for every rational number \( r \).