\section*{3.6. Connectedness}

In this section we undertake a brief study of connectedness in metric spaces. The discerning reader will observe that the flavour of this section is a bit less ‘analytic’ than what has been encountered hitherto.

\textbf{Definition 3.6.1.} Suppose that \((M, \rho)\) is a metric space, and that \(A\) is a subset of \(M\).

(i) We say that \(A\) is \textit{disconnected} if there exist \(\rho\)-open sets \(G\) and \(H\) such that (i) \(A \subseteq G \cup H\), (ii) both \(A \cap G\) and \(A \cap H\) are nonempty, and (iii) \(A \cap G \cap H = \emptyset\). The sets \(G\) and \(H\) are said to provide a \textit{disconnection} for \(A\).

(ii) We say that \(A\) is \textit{connected} if it is not disconnected.

\textbf{Example 3.6.2.} (i) Consider the real line \(\mathbb{R}\) equipped with the usual metric. The set \(\mathbb{Z}\) of integers is a disconnected subset of \(\mathbb{R}\), for it may be verified that the sets \(G = (-1, 1)\) and \(H = (1, \infty)\) provide a disconnection for \(\mathbb{Z}\).

(ii) Consider the normed linear space \(\ell^2\) and let \(A := \{[x_1 \ x_2]^T \in \mathbb{R}^2 : x_1x_2 = 1\}\). The reader is invited to check that the sets \(G = \{[x_1 \ x_2]^T \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}\) and \(H = \{[x_1 \ x_2]^T \in \mathbb{R}^2 : x_1 < 0, x_2 < 0\}\) provide a disconnection for \(A\).

\textbf{Remark 3.6.3.} Let \((M, \rho)\) be a metric space. Confirming the following simple fact is left as a task for the reader: \(M\) is disconnected if there exist nonempty open sets \(G\) and \(H\) such that \(M = G \cup H\) and \(G \cap H = \emptyset\). As before, \(G\) and \(H\) are said to provide a disconnection for \(M\).

Here is an alternate description of disconnected metric spaces.

\textbf{Theorem 3.6.4.} Suppose that \((M, \rho)\) is a metric space. The following are equivalent:

(i) \(M\) is disconnected.

(ii) There exists a nonempty, proper subset of \(M\) which is both \(\rho\)-open and \(\rho\)-closed.

\textbf{Proof.} Suppose that \(M\) is disconnected, and let \(G\) and \(H\) be as in Remark 3.6.3. Now \(G\) is an open set. Furthermore, it is a proper subset of \(M\), because its complement, namely \(H\), is nonempty.

Conversely, if \(S\) is a nonempty, proper subset of \(M\) which is both open and closed, then \(S\) and its complement provide a disconnection for \(M\).

We note in passing that Theorem 3.6.4 may be restated as follows: a metric space \((M, \rho)\) is connected if and only if \(M\) does not contain any nonempty, proper subset which is both open and closed.

\textbf{Example 3.6.5.} Let \(d\) denote the discrete metric on a nonempty set \(M\). The reader will verify that every subset of \(M\) is both \(d\)-open and \(d\)-closed. In particular \((M, d)\) is disconnected.

The section concludes with an elegant description of the structure of connected subsets of the real line.

\textbf{Theorem 3.6.6.} Suppose that \(E\) is a nonempty subset of the real line \(\mathbb{R}\), equipped with the usual metric. The following are equivalent:

(i) \(E\) is connected.

(ii) \(E\) has the following property: if \(x, y \in E\) and \(x < z < y\), then \(z \in E\).

\textbf{Proof.} If \(E\) does not possess the property given in (ii), then there exist \(x, y \in E\) and \(z_0 \notin E\) such that \(x < z_0 < y\). Consider the open sets \(G := (-\infty, z_0)\) and \(H := (z_0, \infty)\). As \(x \in G \cap E\) and
$y \in H \cap E$, neither set is empty. Moreover, $G \cap H \cap E \subseteq G \cap H = \emptyset$, and the fact that $z_0 \notin E$ guarantees that $E \subseteq G \cup H$. Thus $G$ and $H$ provide a disconnection for $E$. This proves that condition (i) implies condition (ii).

The converse is trickier. We show that, if $E$ satisfies condition (ii) and $G$ and $H$ are open sets such that neither $G \cap E$ nor $H \cap E$ is empty but $G \cap H \cap E = \emptyset$, then $E$ cannot be contained in the union of $G$ and $H$. This will show that condition (ii) implies condition (i). Let $G$ and $H$ be as above, and pick $x \in G \cap E$ and $y \in H \cap E$. We may assume without loss that $x < y$. Define $S := \{s \in G \cap E : s < y\}$ and note that it is a nonempty set because it contains $x$. Moreover, $y$ is an upper bound of $S$, so $\alpha := \sup(S)$ is well defined. As $x \in S$ and $\alpha$ is the least upper bound of $S$, we find that $x \leq \alpha \leq y$, whence $\alpha \in E$ by assumption. We shall show that $\alpha$ cannot belong to $G$ or $H$, thereby finishing the proof. Now if $\alpha \in H$, then the openness of $H$ furnishes a positive number $r$ such that $(\alpha - r, \alpha + r) \subseteq H$. Remembering Proposition 2.1.8 along with the definition of $\alpha$ yields an $s_0 \in S$ such that $\alpha - r < s_0 \leq \alpha$. In particular $s_0 \in G \cap E \cap H$, contradicting the fact that this set is empty. Thus $\alpha \notin H$. On the other hand, if $\alpha \in G$, then $\alpha < y$ because $\alpha \leq y$ and $y \notin G$. Combining this with the fact that $G$ is open gives us a positive number $\eta$ such that $(\alpha - \eta, \alpha + \eta) \subseteq G$ and $\alpha + \eta < y$. Now if $t$ is a number such that $\alpha < t < \alpha + \eta$, then $x \leq \alpha < t < y$, and hence $t \in E$ by assumption. Therefore $t \in G \cap E$ and $t < y$, so $t \in S$, violating the fact that $\alpha$ is an upper bound of $S$. Thus $\alpha \notin G$ and the proof is complete.

**Remark 3.6.7.** The reader should now be able to convince herself that the only connected subsets of the real line (equipped with the usual metric) are intervals.

### §3.7. Continuous functions on metric spaces

Having concentrated thus far on various properties of metric spaces, we move now to the treatment of functions that are defined on metric spaces. We begin with an all important definition.

**Definition 3.7.1.** Suppose that $(M, \rho)$ and $(M', \rho')$ are metric spaces, and let $f : M \to M'$ be a function. We say that $f$ is continuous at a point $x_0 \in M$ (or more precisely $\rho - \rho'$ continuous at $x_0$) if the following holds: for every sequence $\{x_n\}$ in $M$ for which $\lim_{n \to \infty} x_n = x_0$, we have $\lim_{n \to \infty} f(x_n) = f(x_0)$ in $M'$. We shall say that $f$ is continuous on $M$ if $f$ is continuous at every point in $M$.

The following characterization of continuity (at a point) should be reminiscent of the well-known ‘$\epsilon - \delta$ definition’ encountered in real analysis.

**Theorem 3.7.2.** Suppose that $(M, \rho)$ and $(M', \rho')$ are metric spaces, and let $f : M \to M'$ be a function. The following are equivalent.

(i) $f$ is continuous at $x_0 \in M$.

(ii) For every $\epsilon > 0$ there is a $\delta > 0$ such that $\rho'(f(x), f(x_0)) < \epsilon$ whenever $\rho(x, x_0) < \delta$.

**Proof.** The failure of condition (ii) means that there is some $\epsilon_0 > 0$ such that, for every $\delta > 0$ there is some $x_\delta \in M$ for which $\rho(x_\delta, x_0) < \delta$ but $\rho'(f(x_\delta), f(x_0)) \geq \epsilon_0$. In particular, for every positive integer $n$ one can find an $x_n \in M$ satisfying the pair of inequalities $\rho(x_n, x_0) < 1/n$ and $\rho'(f(x_n), f(x_0)) \geq \epsilon_0$. The first of these inequalities shows, via the ‘Sandwich Principle’, that $\lim_{n \to \infty} x_n = x_0$ in $M$, whereas the second guarantees that the sequence $\{f(x_n)\}$ does not converge to $f(x_0)$ in $M'$ as $n$ tends to infinity. Therefore $f$ cannot be continuous at $x_0$, and we have proved that condition (i) implies condition (ii).

Assume now that condition (ii) holds. Let us also suppose that $\lim_{n \to \infty} x_n = x_0$ in $M$; we wish to prove that $\lim_{n \to \infty} f(x_n) = f(x_0)$ in $M'$. Let $\epsilon > 0$ be given. Our assumption provides a $\delta > 0$
such that $\rho'(f(x), f(x_0)) < \epsilon$ whenever $\rho(x, x_0) < \delta$. As $\lim_{n \to \infty} x_n = x_0$, there is some positive integer $N$ (which may depend on $\delta$ and hence on $\epsilon$) such that $\rho(x_n, x_0) < \delta$ for every $n \geq N$. It is immediate that $\rho'(f(x_n), f(x_0)) < \epsilon$ for every such $n$ and the proof is complete. 

The following (general) definitions will be of considerable use.

**Definition 3.7.3.** Suppose that $X$ and $Y$ are nonempty sets, and that $f : X \to Y$ is a function. Let $A \subseteq X$ and $B \subseteq Y$. We define

$$f(A) := \{ f(x) : x \in A \} \quad \text{and} \quad f^{-1}(B) := \{ x \in X : f(x) \in B \}.$$  

The first of these sets is called the *image of $A$* whilst the second is referred to as the *inverse image of $B$*.

**Remark 3.7.4.** Let $X$, $Y$, $f$, and $B$ be as above. The reader is asked to verify that the inverse image of the complement of $B$ is the complement of the inverse image of $B$.

The close connexion between continuity and open/closed sets is highlighted in this next result.

**Theorem 3.7.5.** Suppose that $(M, \rho)$ and $(M', \rho')$ are metric spaces, and let $f : M \to M'$ be a function. The following are equivalent.

(i) $f$ is continuous on $M$.
(ii) $f^{-1}(G)$ is $\rho$-open in $M$ whenever $G$ is $\rho'$-open in $M'$.
(iii) $f^{-1}(F)$ is $\rho$-closed in $M$ whenever $F$ is $\rho'$-closed in $M'$.

**Proof.** Assume that $f$ is continuous on $M$ and let $G$ be any open set in $M'$. If $G$ is either the empty set or all of $M'$, then its inverse image is either empty or all of $M$; in either case the inverse image of $G$ is open. Suppose now that $f^{-1}(G)$ is nonempty and let $x_0$ be an arbitrary element in it; the latter condition implies that $f(x_0) \in G$. So the openness of $G$ guarantees an $\epsilon > 0$ such that $U_\rho(f(x_0); \epsilon) \subseteq G$. Now $f$ is assumed to be continuous on $M$, so it must be continuous at $x_0$. Therefore Theorem 3.7.2 supplies a $\delta > 0$ such that $\rho'(f(x), f(x_0)) < \epsilon$ whenever $\rho(x, x_0) < \delta$. In other words, $f(x) \in U_\rho'(f(x_0); \epsilon) \subseteq G$ whenever $x \in U_\rho(x_0; \delta)$, and this in turn implies that $U_\rho(x_0; \delta) \subseteq f^{-1}(G)$. Thus $f^{-1}(G)$ is open in $M$.

We now show that condition (iii) is implied by condition (ii). Let $F$ be a closed set in $M'$, so that $F^c$ (the complement of $F$) is open in $M'$. Therefore $[f^{-1}(F)]^c = f^{-1}(F^c)$ (Remark 3.7.4) is open in $M$ by assumption. Equivalently $f^{-1}(F)$ is closed in $M$.

Finally, we show that the failure of condition (i) implies the failure of condition (iii) (thereby proving that (iii) implies (i) and completing the cycle of implications). If $f$ is not continuous on $M$ then it is not continuous at some $x_0$ in $M$. So there is some sequence $\{x_n\} \in M$ such that $\lim_{n \to \infty} x_n = x_0$ but $\{f(x_n)\}$ does not converge to $f(x_0)$. The latter condition implies the existence of an $\eta > 0$ and an increasing sequence of positive integers, say $n_1 < n_2 < \cdots$, such that $\rho'(f(x_{n_k}), f(x_0)) \geq \eta$ for every positive integer $k$. Hence $x_{n_k} \in f^{-1}([U_\rho'(f(x_0); \eta)]^c)$ for every $k \in \mathbb{N}$. As $\lim_{n \to \infty} x_n = x_0$ (every subsequence of a convergent sequence also converges to the limit of the original sequence) and $x_0$ is patently outside $f^{-1}([U_\rho'(f(x_0); \eta)]^c) = [f^{-1}(U_\rho'(f(x_0); \eta))]^c$ (because $x_0$ belongs to $f^{-1}(U_\rho'(f(x_0); \eta)))$, the set $f^{-1}([U_\rho'(f(x_0); \eta)]^c)$ cannot be closed (thanks to Theorem 3.2.9). Thus we stand in violation of condition (iii), because $[U_\rho'(f(x_0); \eta)]^c$ is a closed set (via Theorem 3.2.5). 

The remainder of this section will be devoted to exploring the action of continuous functions on connected sets and compact sets. In particular, we shall discover that both properties are preserved under continuous maps, an altogether satisfactory state of affairs indeed.
Theorem 3.7.6. Suppose that \((M, \rho)\) and \((M', \rho')\) are metric spaces, and let \(f : M \to M'\) be a function. If \(f\) is continuous on \(M\) and \(A\) is a connected subset of \(M\), then \(f(A)\) is a connected subset of \(M'\).

Proof. If \(f(A)\) is disconnected, then there exist \(\rho'\)-open sets \(G\) and \(H\) which provide a disconnection for \(f(A)\) in \(M'\). It is a fairly simple matter to check that the sets \(f^{-1}(G)\) and \(f^{-1}(H)\) – which are \(\rho\)-open because of Theorem 3.7.5 – provide a disconnection for \(A\) in \(M\). It follows that if \(A\) is connected, then so is \(f(A)\).

As a direct consequence of the foregoing theorem one obtains a (modest) generalization of the familiar Intermediate Value Theorem.

Corollary 3.7.8. (Generalized Intermediate Value Theorem) Suppose that \((M, \rho)\) is a connected metric space, and that the real line \(\mathbb{R}\) is equipped with the usual metric. Let \(f : M \to \mathbb{R}\) be continuous on \(M\). If \(a, b \in M\) and \(\gamma\) is a real number between \(f(a)\) and \(f(b)\), then there is some \(c \in M\) such that \(f(c) = \gamma\).

Proof. EXERCISE (Use Theorem 3.7.6 along with Remark 3.6.7).

The reader will also convince herself that the standard Intermediate Value Theorem is, in fact, a consequence of Corollary 3.7.8.

Theorem 3.7.9. Suppose that \((M, \rho)\) and \((M', \rho')\) are metric spaces, and let \(f : M \to M'\) be a function. If \(f\) is continuous on \(M\) and \(A\) is a compact subset of \(M\), then \(f(A)\) is a compact subset of \(M'\).

Proof. On account of Theorem 3.5.4, it suffices to show the following: if \(\{b_n\}\) is any sequence in \(f(A)\) then there is a subsequence \(\{b_{n_k} : k \in \mathbb{N}\}\) of \(\{b_n\}\) as well as an element \(b \in f(A)\) such that \(\lim_{k \to \infty} b_{n_k} = b\). Let \(b_n = f(a_n), a_n \in A, n \in \mathbb{N}\). The compactness of \(A\) yields a subsequence \(\{a_{n_k} : k \in \mathbb{N}\}\) of \(\{a_n\}\) as well as an element \(a \in A\) such that \(\lim_{k \to \infty} a_{n_k} = a\). As \(f\) is continuous at \(a\) and \(\lim_{k \to \infty} f(a_{n_k}) = f(a)\), we must have \(\lim_{k \to \infty} f(a_{n_k}) = f(a)\). Choosing \(b = f(a)\) finishes the proof.

Specializing to real-valued functions one obtains the following.

Theorem 3.7.10. Suppose that \((M, \rho)\) is a metric space, and that the real line \(\mathbb{R}\) is equipped with the usual metric. Let \(f : M \to \mathbb{R}\) be continuous on \(M\), and let \(A\) be a compact subset of \(M\). The following hold:

(i) \(f(A)\) is a closed and bounded subset of \(\mathbb{R}\).

(ii) Let \(\alpha^* := \sup \{f(a) : a \in A\}\) and \(\alpha_* := \inf \{f(a) : a \in A\}\). There exist elements \(a^*\) and \(a_*\) in \(A\) such that \(f(a^*) = \alpha^*\) and \(f(a_*) = \alpha_*\).

Proof. The first assertion follows at once from Theorem 3.7.9 and the Heine-Borel Theorem.

The set \(f(A)\) being bounded, it is contained in the interval \([-T, T]\) for some \(T > 0\) (Lemma 3.4.7). In particular \(\alpha^*\) and \(\alpha_*\) are well defined. Let \(n\) be a(ny) positive integer. Proposition 2.1.8 provides an \(a_n \in A\) such that \(\alpha^* - (1/n) < f(a_n) \leq \alpha^*\), whence

\[
\lim_{n \to \infty} f(a_n) = \alpha^*.
\]  

Now the compactness of \(A\) yields a subsequence \(\{a_{n_k} : k \in \mathbb{N}\}\) of \(\{a_n\}\) as well as an element \(a^* \in A\) such that \(\lim_{k \to \infty} a_{n_k} = a^*\). As \(f\) is continuous at \(a^*\) and \(\lim_{k \to \infty} f(a_{n_k}) = f(a^*)\), we must have

\[
\lim_{k \to \infty} f(a_{n_k}) = f(a^*).
\]
From (3.7.1) and (3.7.2) we find that $\alpha^* = f(a^*)$. The argument involving $\alpha_*$ is similar.

We wish to close the section with an important example, but some groundwork must be laid first. Let $M$ be a nonempty set; suppose that $f$ and $g$ are functions from $M$ into the real line $\mathbb{R}$, and that $\alpha$ is a real number. We define the functions $f + g$, $fg$, $\alpha f$, and $|f|$ in the expected way: for every $x \in M$,

$$(f + g)(x) := f(x) + g(x), \quad (fg)(x) := f(x)g(x), \quad (\alpha f)(x) := \alpha f(x), \quad \text{and} \quad |f|(x) := |f(x)|.$$ 

The following result is basic to further development.

**Theorem 3.7.11.** Suppose that $(M, \rho)$ is a metric space, and that the real line $\mathbb{R}$ is equipped with the usual metric. Let $f$ and $g$ be functions from $M$ into $\mathbb{R}$, and let $x_0$ be a fixed element in $M$. The following hold:

(i) If $f$ and $g$ are continuous at $x_0$, then so is $f + g$.
(ii) If $f$ is continuous at $x_0$ and $\alpha$ is any fixed real number, then $\alpha f$ is also continuous at $x_0$.
(iii) If $f$ is continuous at $x_0$, then so is $|f|$.
(iv) If $f$ and $g$ are continuous at $x_0$, then so is $fg$.

**Proof.** EXERCISE.

The following example, which will bring this section to a close, invites an obvious comparison with Example 3.1.5(iv).

**Example 3.7.12.** Suppose that $(K, \rho)$ is a compact metric space, and let the real line $\mathbb{R}$ be equipped with the usual metric. We define

$$C(K) := \{f : K \to \mathbb{R} : f \text{ is continuous on } K\}.$$ 

No doubt the reader will be delighted at being afforded the opportunity to verify the following facts: (i) $C(K)$, when endowed with the operations of addition and scalar multiplication discussed above, is a vector space over $\mathbb{R}$; (ii) $|f| \in C(K)$ whenever $f \in C(K)$, and (iii) if $f, g \in C(K)$, then $fg$ belongs to $C(K)$ as well. The reader will further prolong her pleasure by also proving that the function

$$\|f\|_{\infty} := \sup\{|f(x)| : x \in K\}, \quad f \in C(K),$$

is a norm on $C(K)$ (usually called the *supremum norm* or the *uniform norm*), and that

$$\|fg\|_{\infty} \leq \|f\|_{\infty} \|g\|_{\infty}, \quad f, g \in C(K).$$