APPENDIX 2. GENERALIZED MINKOWSKI INEQUALITY

Theorem A2.1. *(Generalized Minkowski Inequality) Under appropriate conditions (on the function h which appears below), and for $1 \le p < \infty$, the following inequality holds:*

$$\left[\int_a^b \left|\int_c^d h(x,y)\,dy\right|^p dx\right]^{1/p} \le \int_c^d \left[\int_a^b |h(x,y)|^p\,dx\right]^{1/p} dy\,.$$

Proof. The case $p = 1$ follows from Fubini's Theorem, so we assume that $p > 1$ and note the following:

$$\begin{aligned}
\int_a^b \left|\int_c^d h(x,y)\,dy\right|^p dx &= \int_a^b \left|\int_c^d h(x,y)\,dy\right|^{p-1} \left|\int_c^d h(x,y)\,dy\right| dx \\
&\le \int_a^b \left|\int_c^d h(x,t)\,dt\right|^{p-1} \left[\int_c^d |h(x,y)|\,dy\right] dx \\
&= \int_a^b \left[\int_c^d \left|\int_c^d h(x,t)\,dt\right|^{p-1} |h(x,y)|\,dy\right] dx \\
&= \int_c^d \left[\int_a^b \left|\int_c^d h(x,t)\,dt\right|^{p-1} |h(x,y)|\,dx\right] dy\,, \qquad \text{(A2.1)}
\end{aligned}$$

the final step coming from Fubini's Theorem. Let $q := p/(p-1)$ and apply Hölder's Inequality to the inner integral (with respect to x) in (A2.1):

$$\begin{aligned}
\int_a^b \left|\int_c^d h(x,t)\,dt\right|^{p-1} |h(x,y)|\,dx &\le \left[\int_a^b \left|\int_c^d h(x,t)\,dt\right|^{q(p-1)} dx\right]^{1/q} \left[\int_a^b |h(x,y)|^p\,dx\right]^{1/p} \\
&= \left[\int_a^b \left|\int_c^d h(x,t)\,dt\right|^{p} dx\right]^{1/q} \left[\int_a^b |h(x,y)|^p\,dx\right]^{1/p} . \qquad \text{(A2.2)}
\end{aligned}$$

Using (A2.2) in (A2.1) and noting that the first factor on the right-hand side of (A2.2) is a number yields the relations

$$\begin{aligned}
\int_a^b \left|\int_c^d h(x,y)\,dy\right|^p dx &\le \int_c^d \left[\left[\int_a^b \left|\int_c^d h(x,t)\,dt\right|^{p} dx\right]^{1/q} \left[\int_a^b |h(x,y)|^p\,dx\right]^{1/p}\right] dy \\
&= \left[\int_a^b \left|\int_c^d h(x,t)\,dt\right|^{p} dx\right]^{1/q} \int_c^d \left[\int_a^b |h(x,y)|^p\,dx\right]^{1/p} dy\,.
\end{aligned}$$

Dividing both sides by the number $\left[\int_a^b \left|\int_c^d h(x,t)\,dt\right|^p dx\right]^{1/q}$ and remembering that $1-(1/q)=1/p$ gives the required inequality. ∎

Reference

[Zy] Zygmund, A., *Trigonometric Series, Volume I*, Cambridge University Press, 1968, pp. 18-19.