APPENDIX 3

Lemma A3.1. *If α and β are nonnegative real numbers and $1 \le p < \infty$, then* $(\alpha+\beta)^p \le 2^{p-1}(\alpha^p+\beta^p)$.

Proof. The result being obvious if $\alpha = \beta = 0$, we assume that at least one of the numbers, say α, is positive. The required inequality is equivalent to the following: $\left(1+\frac{\beta}{\alpha}\right)^p \le 2^{p-1}\left(1+\frac{\beta^p}{\alpha^p}\right)$. So it suffices to prove that $(1+x)^p \le 2^{p-1}(1+x^p)$ for every $x \ge 0$. As this is apparent for $p = 1$, we assume that $p > 1$. Consider the function $H(x) := (1+x)^p/(1+x^p)$, $x \ge 0$. Basic calculus shows that $\max\{H(x) : 0 \le x < \infty\} = H(1) = 2^{p-1}$, whence the result. ∎

In what follows we assume that a and b are finite real numbers with $a < b$.

Theorem A3.2. *The space $C[a,b]$ is dense in $L^p[a,b]$ for any $1 \le p < \infty$.*

Proof. Let $f \in L^p[a,b]$ and let $\epsilon > 0$ be given. Define, for every positive integer n, the function

$$f_n(x) := \begin{cases} f(x), & \text{if } |f(x)| \le n; \\ n, & \text{if } |f(x)| > n. \end{cases}$$

Then $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for every x such that $|f(x)| < \infty$. As $f \in L^p[a,b]$, f must be finite almost everywhere, so $\{f_n\}$ converges to f almost everywhere. As $|f_n(x)| \le |f(x)|$ for almost all x, the triangle inequality provides the relations $|f_n(x) - f(x)|^p \le (|f_n(x)| + |f(x)|)^p \le 2^p|f(x)|^p$ for almost every x. This, taken together with the fact that $f \in L^p[a,b]$, implies, via the Dominated Convergence Theorem [HS, (12.30)Theorem, page 174], that $\lim\limits_{n\to\infty} \int_a^b |f_n - f|^p = 0$. Choose a positive integer N such that $\int_a^b |f_N - f|^p < (\epsilon/2)^p$. Lusin's Theorem [HS, page 159] supplies a continuous function g such that

$$\sup\{|g(x)| : a \le x \le b\} \le \sup\{|f_N(x)| : a \le x \le b\} \le N,$$

and $m\{x \in [a,b] : f_N(x) \ne g(x)\} < \left(\frac{\epsilon}{4N}\right)^p$ (where m denotes Lebesgue measure); put $E := \{x \in [a,b] : f_N(x) \ne g(x)\}$. Combining these facts with Minkowski's inequality, the triangle inequality, and Lemma A3.1 yields the following chain of relations:

$$\begin{aligned}
\left[\int_a^b |f-g|^p\right]^{1/p} &\le \left[\int_a^b |f-f_N|^p\right]^{1/p} + \left[\int_a^b |f_N-g|^p\right]^{1/p} \\
&< \frac{\epsilon}{2} + \left[\int_E |f_N-g|^p\right]^{1/p} \\
&\le \frac{\epsilon}{2} + \left[\int_E 2^{p-1}(|f_N|^p + |g|^p)\right]^{1/p} \\
&\le \frac{\epsilon}{2} + \left[2^p N^p m(E)\right]^{1/p} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,.
\end{aligned}$$

This completes the proof. ∎

Theorem A3.3. *If* $1 \le p < \infty$ *and* $f \in L^p(\mathbf{T})$, *then* $\lim_{h\to 0} \int_{-\pi}^{\pi} |f(t+h) - f(t)|^p \, dt = 0$.

Proof. We assume without loss of generality that $|h| \le 1$. As $f \in L^p(\mathbf{T})$, f belongs to $L^p[-\pi-1, \pi+1]$. Let $\epsilon > 0$ be given. Theorem A3.2 furnishes a function g which is continuous on $[-\pi-1, \pi+1]$ and

$$\left[\int_{-\pi-1}^{\pi+1} |f(t) - g(t)|^p \, dt\right]^{1/p} < \frac{\epsilon}{3}. \tag{A3.1}$$

From Minkowski's Inequality we find that

$$\left[\int_{-\pi}^{\pi} |f(t+h) - f(t)|^p \, dt\right]^{1/p} \le \left[\int_{-\pi}^{\pi} |f(t+h) - g(t+h)|^p \, dt\right]^{1/p} + \left[\int_{-\pi}^{\pi} |g(t+h) - g(t)|^p \, dt\right]^{1/p}$$
$$+ \left[\int_{-\pi}^{\pi} |g(t) - f(t)|^p \, dt\right]^{1/p}. \tag{A3.2}$$

The third summand on the right is no bigger than the left-hand side of (A3.1), hence it is smaller than $\epsilon/3$. The left-hand side of (A3.1) also dominates the first summand on the right-hand side of (A3.2), because $|h| \le 1$. Thus (A3.2) can be estimated as follows:

$$\left[\int_{-\pi}^{\pi} |f(t+h) - f(t)|^p \, dt\right]^{1/p} < \frac{2\epsilon}{3} + \left[\int_{-\pi}^{\pi} |g(t+h) - g(t)|^p \, dt\right]^{1/p}. \tag{A3.3}$$

As g is uniformly continuous on $[-\pi-1, \pi+1]$, there is a $\delta \in (0,1)$ such that $|g(u) - g(v)| < \frac{\epsilon}{3(2\pi)^{1/p}}$ whenever $-\pi - 1 \le u, v \le \pi + 1$ and $|u - v| \le \delta$. Therefore

$$\int_{-\pi}^{\pi} |g(t+h) - g(t)|^p \, dt \le \frac{\epsilon^p}{3^p}, \qquad |h| < \delta. \tag{A3.4}$$

Combining (A3.3) and (A3.4) yields

$$\left[\int_{-\pi}^{\pi} |f(t+h) - f(t)|^p \, dt\right]^{1/p} < \epsilon, \qquad |h| < \delta,$$

thereby finishing the proof. ∎

Reference

[HS] Hewitt, E. and K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, 1969.