APPENDIX 4. FROM FEJÉR TO WEIERSTRAß

The purpose of this section is to show how Fejér's Theorem (Theorem 1.2.9(ii)) can be used to derive the famous Weierstraß Polynomial Approximation Theorem.

Lemma A4.1. *Suppose that $F \in C(\mathbf{T})$ is real valued and even. The following hold:*

(i) Every Fourier coefficient of F is a real number.

(ii) $F[-k] = F[k]$ for every positive integer k.

(iii) If N is any nonnegative integer, then $\sigma_N[F,t]$ is a polynomial of the form $\sum_{k=0}^{N} A_k \cos(kt)$, where the A_k's are real numbers.

Proof. (i) If $n = 0$, then

$$F[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)\,dt \in \mathbf{R}.$$

Suppose now that $n \neq 0$. Then

$$\begin{aligned} F[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)e^{-int}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)\cos(nt)\,dt - \frac{i}{2\pi}\int_{-\pi}^{\pi} F(t)\sin(nt)\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)\cos(nt)\,dt \in \mathbf{R}, \end{aligned} \tag{A4.1}$$

the final equation stemming from the fact that the function $t \mapsto F(t)\sin(nt)$ is odd.

(ii) This is immediate from (A4.1) because $\cos(-kt) = \cos(kt)$.

(iii) Recall that $\sigma_N[F,t] = \sum_{k=-N}^{N} \alpha_k e^{ikt}$, where

$$\alpha_k := \left(1 - \frac{|k|}{N+1}\right) F[k], \qquad -N \le k \le N.$$

Assertions (i) and (ii) of the current lemma imply that α_k is a real number for every $|k| \le N$ and $\alpha_{-k} = \alpha_k$ for every positive integer k. Therefore

$$\begin{aligned} \sigma_N[F,t] &= \alpha_0 + \sum_{k=1}^{N} \alpha_k[\cos(kt) + i\sin(kt)] + \sum_{k=-N}^{-1} \alpha_k[\cos(kt) + i\sin(kt)] \\ &= \alpha_0 + \sum_{k=1}^{N} (\alpha_k + \alpha_{-k})\cos(kt) + i\sum_{k=1}^{N} (\alpha_k - \alpha_{-k})\sin(kt) \\ &= \alpha_0 + 2\sum_{k=1}^{N} \alpha_k \cos(kt). \end{aligned}$$

∎

Lemma A4.2. *Every trigonometric polynomial of the form*

$$p(t) = \sum_{k=0}^{N} A_k \cos(kt), \qquad A_k \in \mathbf{R}, \quad 0 \le k \le N,$$

admits the following representation:

$$p(t) = \sum_{k=0}^{N} B_k \cos^k t, \qquad B_k \in \mathbf{R}, \quad 0 \le k \le N.$$

Proof. It suffices to show that, for every positive integer m, the function $\cos(mt)$ is a real linear combination of the elements of the set $\{\cos^l t \,:\, 0 \le l \le m\}$. The DeMoivre and Binomial Theorems combine to yield the identities

$$\cos(mt) + i\sin(mt) = (\cos t + i \sin t)^m = \sum_{k=0}^{m} \binom{m}{k} i^k \sin^k t \cos^{m-k} t, \quad t \in \mathbf{R}.$$

Comparing the real parts of the first and third terms above, one finds that

$$\cos(mt) = \sum_{r=0}^{\lfloor m/2 \rfloor} \binom{m}{2r} i^{2r} \sin^{2r} t \cos^{m-2r} t = \sum_{r=0}^{\lfloor m/2 \rfloor} \binom{m}{2r} (-1)^r (1-\cos^2 t)^r \cos^{m-2r} t, \quad t \in \mathbf{R},$$

and the last expression is a real linear combination of the functions $1, \cos t, \ldots, \cos^m t$. ∎

We are now ready for Weierstraß's theorem, which will be given in two steps. The first (and key) step is the following:

Theorem A4.3. *(Weierstraß) Suppose that* $g : [-1,1] \to \mathbf{R}$ *is continuous on* $[-1,1]$. *Given* $\epsilon > 0$, *there is a polynomial* $s(x) = \sum_{k=0}^{N} B_k x^k$, $B_k \in \mathbf{R}$, $0 \le k \le N$, *such that* $|g(x) - s(x)| < \epsilon$ *for every* $x \in [-1,1]$.

Proof. Define $F(t) := g(\cos t)$, which is a real-valued, even function belonging to $C(\mathbf{T})$. Thanks to Fejér's Theorem, Lemma A4.1(iii), and Lemma A4.2, there is a nonnegative integer N and real numbers $B_0, \ldots, B_N$ such that

$$\left| g(\cos t) - \sum_{k=0}^{N} B_k \cos^k t \right| = \left| F(t) - \sum_{k=0}^{N} B_k \cos^k t \right| < \epsilon \quad \text{for every } t \in [0, \pi]. \tag{A4.2}$$

Putting $x = \cos t$ and $s(x) = \sum_{k=0}^{N} B_k x^k$, we find that (A4.2) is equivalent to

$$|g(x) - s(x)| < \epsilon \quad \text{for every } x \in [-1,1].$$

∎

The general form of Weierstraß's Theorem is derived from Theorem A4.3 as follows:

Theorem A4.4. *(Weierstraß's Polynomial Approximation Theorem) Let* a *and* b *be a pair of fixed real numbers with* $a < b$. *Suppose that* $f : [a,b] \to \mathbf{R}$ *is continuous on* $[a,b]$. *Given* $\epsilon > 0$, *there is a polynomial* P *such that* $|f(y) - P(y)| < \epsilon$ *for every* $y \in [a,b]$.

Proof. Define $g(x) = f\left(a + \frac{(b-a)(x+1)}{2}\right)$, $-1 \le x \le 1$. Then g is a continuous real-valued function on the interval $[-1,1]$. Accordingly, Theorem A4.3 supplies a polynomial s such that

$$\left| f\left(a + \frac{(b-a)(x+1)}{2}\right) - s(x) \right| = |g(x) - s(x)| < \epsilon \quad \text{for every } x \in [-1,1]. \tag{A4.3}$$

Putting $y := a + \frac{(b-a)(x+1)}{2}$ and $P(y) := s\left(\frac{2(y-a)}{b-a} - 1\right)$, one finds that (A4.3) may be rewritten as

$$|f(y) - P(y)| < \epsilon \quad \text{for every } y \in [a,b]\,.$$

As P is the composition of the polynomial s with the linear function $y \mapsto \frac{2(y-a)}{b-a} - 1$, P itself is a polynomial in the variable y. ∎

EXERCISE: Use induction to show that, for every positive integer m, the function $\cos(mt)$ is a real linear combination of the elements of the set $\{\cos^l t \,:\, 0 \le l \le m\}$.