CHAPTER 2. FOURIER TRANSFORMS I

§2.1. Preliminaries

In this chapter we consider the notion of Fourier transforms for integrable functions on the real line. This is an aperiodic analogue of the theory developed in the previous chapter.

Definition 2.1.1. Suppose that \( f \in L^1(\mathbb{R}) \). The Fourier transform of \( f \) is defined as follows:

\[
\hat{f}(\gamma) := \int_{-\infty}^{\infty} f(t) e^{-i\gamma t} \, dt, \quad \gamma \in \mathbb{R}.
\]

The integral above is well defined for every real number \( \gamma \), because \( f \), and hence the function \( t \mapsto f(t)e^{-i\gamma t} \), is integrable on \((-\infty, \infty)\), for every fixed \( \gamma \). Thus the Fourier transform of an integrable function is a complex-valued function defined on \( \mathbb{R} \).

Some simple properties of the Fourier transform are listed below:

**Theorem 2.1.2.** Let \( f, g \in L^1(\mathbb{R}) \). The following hold:

(i) Let \( c_1 \) and \( c_2 \) be complex numbers, and define \( H := c_1 f + c_2 g \). Then \( \hat{H}(\gamma) = c_1 \hat{f}(\gamma) + c_2 \hat{g}(\gamma) \) for every real number \( \gamma \).

(ii) If \( \overline{f} \) denotes the complex conjugate of \( f \), then \( \hat{\overline{f}}(\gamma) = \overline{\hat{f}(-\gamma)} \) for every real number \( \gamma \).

(iii) Let \( u \) be a real number, and define \((\tau_u f)(\cdot) := f(\cdot - u)\). Then \((\tau_{-u} f)(\gamma) = e^{-iu\gamma} \hat{f}(\gamma), \gamma \in \mathbb{R}\).

(iv) Let \( \lambda \in \mathbb{R} \), and define \( e^{\lambda t}(t) := e^{i\lambda t}, t \in \mathbb{R} \). Then \((e^{\lambda t} f)(\gamma) = \hat{f}(\gamma - \lambda), \gamma \in \mathbb{R}\).

(v) Let \( \lambda \) be a fixed nonzero real number, and define the \( \lambda \)-dilation of \( f \) by \( f_\lambda(\cdot) := \lambda f(\lambda \cdot) \). Then \( \hat{f}_\lambda(\gamma) = \frac{1}{|\lambda|} \hat{f}(\frac{\gamma}{\lambda}), \gamma \in \mathbb{R}\).

**Proof.** EXERCISE.

The next result records some analytic properties of the Fourier transform.

**Theorem 2.1.3.** Suppose that \( f \in L^1(\mathbb{R}) \). The following hold:

(i) \( |\hat{f}(\gamma)| \leq \|f\|_{L^1(\mathbb{R})} \) for every real number \( \gamma \).

(ii) The function \( \gamma \mapsto \hat{f}(\gamma) \) is uniformly continuous on \((-\infty, \infty)\).

(iii) (Riemann-Lebesgue Lemma for Fourier Transforms) \( \lim_{|\gamma| \to \infty} \hat{f}(\gamma) = 0 \).

**Proof.** (i)

\[
|\hat{f}(\gamma)| = \left| \int_{-\infty}^{\infty} f(t) e^{-i\gamma t} \, dt \right| \leq \int_{-\infty}^{\infty} |f(t)| \, dt = \|f\|_{L^1(\mathbb{R})}.
\]

(ii) Suppose \( \gamma \) and \( h \) are real numbers. The relation

\[
\hat{f}(\gamma + h) - \hat{f}(\gamma) = \int_{-\infty}^{\infty} f(t) e^{-i\gamma t} [e^{-ih t} - 1] \, dt
\]
leads to the inequality
\[
\left| \hat{f}(\gamma + h) - \hat{f}(\gamma) \right| \leq \int_{-\infty}^{\infty} |f(t)||e^{-iht} - 1| dt.
\]
As the integrand on the right-hand side of the foregoing inequality converges to zero as \( h \) tends to zero (independently of \( \gamma \)), and is no larger than \( 2|f(t)| \) for every \( t \) and \( h \), the required result follows from the Dominated Convergence Theorem.

(iii) Let \( \gamma \in \mathbb{R} \setminus \{0\} \). The relations

\[
\hat{f}(\gamma) = \int_{-\infty}^{\infty} f(t)e^{-i\gamma t} dt = \int_{-\infty}^{\infty} f(t)e^{-i\gamma (t - \frac{\pi}{\gamma})} dt = -\int_{-\infty}^{\infty} f\left(t + \frac{\pi}{\gamma}\right) e^{-i\gamma t} dt
\]

imply that

\[
2\hat{f}(\gamma) = \int_{-\infty}^{\infty} \left[f(t) - f\left(t + \frac{\pi}{\gamma}\right)\right] e^{-i\gamma t} dt,
\]
whence

\[
|2\hat{f}(\gamma)| \leq \int_{-\infty}^{\infty} \left|f(t) - f\left(t + \frac{\pi}{\gamma}\right)\right| dt.
\]

The right-hand side of the preceding inequality approaches zero as \( \gamma \to \pm \infty \), via a well-known result in analysis; see, for example, [HS, page 199].

We now wish to examine the interaction between differentiation and Fourier transforms. The following lemma fulfills a preludial rôle.

**Lemma 2.1.4.** Suppose that \( f \in L^1(\mathbb{R}) \), and that it satisfies each of the following conditions:

(i) \( f \) is absolutely continuous on every closed and bounded subinterval of \( \mathbb{R} \), and (ii) \( f' \in L^1(\mathbb{R}) \).

Then \( \lim_{|t| \to \infty} f(t) = 0 \).

**Proof.** We consider the case \( t \to \infty \); the other case is entirely analogous. If \( \lim_{t \to \infty} f(t) \neq 0 \), then there is a positive number \( \eta \), and a sequence of positive numbers, \( \{t_n : n \in \mathbb{N}\} \), such that \( t_n < t_{n+1} \) for every \( n \), \( \lim_{n \to \infty} t_n = \infty \), and \( |f(t_n)| \geq 2\eta \) for every \( n \). As \( f' \) is integrable on \( \mathbb{R} \), the Dominated Convergence Theorem asserts that \( \lim_{N \to \infty} \int_{-\infty}^{N} |f'(t)| dt = \int_{-\infty}^{\infty} |f'|. \) Consequently, there is some positive number \( M \) such that \( \int_{|t| > M} |f'(t)| dt < \eta \). Choose a positive integer \( k \) such that \( t_k > M \). If \( x > t_k \), then the absolute continuity of \( f \) in the interval \( [t_k, x] \) implies that

\[
|f(x) - f(t_k)| = \left| \int_{t_k}^{x} f'(t) dt \right| \leq \int_{M}^{\infty} |f'(t)| dt < \eta,
\]
whence \( |f(x)| \geq |f(t_k)| - |f(x) - f(t_k)| > \eta \) for every \( x > t_k \). But this contradicts the absolute integrability of \( f \) on \( (-\infty, \infty) \), and the proof is complete.

Theorem 2.1.5. (i) Suppose that $f$ satisfies all the hypotheses of the preceding lemma. Then 
$$(\hat{f}'(\gamma)) = i\gamma \hat{f}(\gamma)$$
for every real number $\gamma$.
(ii) Suppose that each of the functions $t \mapsto f(t)$ and $t \mapsto (-it)f(t) =: H(t)$ belongs to $L^1(\mathbb{R})$. Then 
$$(\hat{f}'(\gamma)) = \hat{H}(\gamma), \gamma \in \mathbb{R}.$$  
Proof. (i) Let $R$ be a large positive number. Integration by parts provides the equation

$$\int_{-R}^{R} f'(t)e^{-it\gamma} dt = f(R)e^{-i\gamma R} - f(-R)e^{i\gamma R} + i\gamma \int_{-R}^{R} f(t)e^{-it\gamma} dt.$$  
(2.1.1)

The first two terms on the right-hand side of (2.1.1) approach zero as $R$ tends to infinity, via Lemma 2.1.4, and applying the Dominated Convergence Theorem to (2.1.1) yields

$$\hat{(f')}(\gamma) = \int_{-\infty}^{\infty} f'(t)e^{-it\gamma} dt = \lim_{R \to \infty} \int_{-R}^{R} f'(t)e^{-it\gamma} dt = \hat{f}(\gamma) \lim_{R \to \infty} \int_{-R}^{R} f(t)e^{-it\gamma} dt = i\gamma \int_{-\infty}^{\infty} f(t)e^{-it\gamma} dt = i\gamma \hat{f}(\gamma).$$

(ii) Let $\gamma$ and $h$ be real numbers, with $h \neq 0$. We have the equation

$$\frac{\hat{f}(\gamma + h) - \hat{f}(\gamma)}{h} = \int_{-\infty}^{\infty} f(t)e^{-it\gamma} \left[ e^{-ih\gamma t} - 1 \right] dt.$$  
(2.1.2)

As $|e^{i\theta}| = 1$, and $|e^{i\theta} - 1| = 2|\sin(\theta/2)| \leq |\theta|$ for every real number $\theta$, we find that the integrand in (2.1.2) is bounded in modulus by $|t f(t)|$ for every $t, \gamma \in \mathbb{R}$ and every $h \in \mathbb{R} \setminus \{0\}$. Moreover, $\lim_{h \to 0} e^{-ih\gamma t} = -it$ for every real number $t$, so the integrability of the function $t \mapsto tf(t)$ allows us, via the Dominated Convergence Theorem, to conclude from (2.1.2) that

$$\lim_{h \to 0} \frac{\hat{f}(\gamma + h) - \hat{f}(\gamma)}{h} = \int_{-\infty}^{\infty} (-it)f(t)e^{-it\gamma} dt, \quad \gamma \in \mathbb{R}.$$  

In other words, $(\hat{f}')(\gamma) = \hat{H}(\gamma), \gamma \in \mathbb{R}$.  

Remark 2.1.6. Expected generalizations of Theorem 2.1.5 may be obtained under suitable hypotheses; if $l$ is any positive integer and $H(t) := (-it)^lf(t)$, then $(\hat{f}^{(l)})(\gamma) = (i\gamma)^l\hat{f}(\gamma)$, and $(\hat{f}^{(l)})(\gamma) = \hat{H}(\gamma)$ for every real number $\gamma$.

Example 2.1.7. We conclude the section by computing the Fourier transform of the Gaussian function $G(t) := e^{-t^2}, t \in \mathbb{R}$, in three different ways. We shall need the following well-known fact; see, for example, [M, page 272].*

$$\hat{G}(0) = \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}.$$  
(2.1.3)

As $G$ is an even function, we obtain
\[
\hat{G}(\gamma) = \int\limits_{-\infty}^{\infty} G(t)e^{-i\gamma t} \, dt = \int\limits_{-\infty}^{\infty} G(t)e^{i\gamma t} \, dt = \hat{G}(-\gamma), \quad \gamma \in \mathbb{R}, \tag{2.1.4}
\]
so it suffices to consider $\gamma > 0$. We begin by writing
\[
\hat{G}(\gamma) = \int\limits_{-\infty}^{\infty} e^{-t^2} e^{-i\gamma t} \, dt = e^{-\frac{\gamma^2}{4}} \int\limits_{-\infty}^{\infty} e^{-(t+\frac{i\gamma}{2})^2} \, dt =: e^{-\frac{\gamma^2}{4}} I(\gamma), \tag{2.1.5}
\]
and turn our attention to computing $I(\gamma)$. Let $R$ and $S$ be positive numbers, and let $C$ be the positively-oriented contour in the complex plane comprising the following four pieces: $C_1$, the horizontal line segment from the point $z = -R$ to the point $z = S$, followed by $C_2$, the vertical line segment from $z = S$ to $z = S+i(\gamma/2)$, followed by the horizontal line segment $C_3$ from $z = S+i(\gamma/2)$ to $z = -R+i(\gamma/2)$, and finally the vertical line segment $C_4$ from $z = -R+i(\gamma/2)$ to $z = -R$. Let $E(z) := e^{-z^2}$, $z \in \mathbb{C}$; Cauchy’s Theorem proclaims that $\int\limits_{C} E(z) \, dz = 0$. Parametrizing each of the four line segments which constitute $C$, we obtain
\[
\int\limits_{-R}^{S} e^{-t^2} \, dt + i \int\limits_{0}^{\gamma/2} e^{-(S+it)^2} \, dt = \int\limits_{-R}^{S} e^{-(t+\frac{i\gamma}{2})^2} \, dt - i \int\limits_{0}^{\gamma/2} e^{-(S+it)^2} \, dt = 0,
\]
whence
\[
\int\limits_{-R}^{S} e^{-(t+\frac{i\gamma}{2})^2} \, dt = \int\limits_{-R}^{S} e^{-t^2} \, dt + i \int\limits_{0}^{\gamma/2} e^{-(S+it)^2} \, dt - i \int\limits_{0}^{\gamma/2} e^{-(S+it)^2} \, dt =: I_1 + I_2 + I_3. \tag{2.1.6}
\]
If $0 \leq t \leq \gamma/2$, then $|e^{-(S+it)^2}| = e^{-S^2+t^2} \leq e^{-S^2+\frac{\gamma^2}{4}}$, so $|I_2| \leq \frac{\gamma}{2} e^{-S^2+\frac{\gamma^2}{4}}$. As the right-hand side of this inequality approaches zero as $S$ tends to infinity, we deduce that $\lim\limits_{S \to \infty} I_2 = 0$. Likewise $\lim\limits_{R \to \infty} I_3 = 0$, so we find from (2.1.6) and (2.1.3) that
\[
I(\gamma) = \lim\limits_{R,S \to \infty} \int\limits_{-R}^{S} e^{-(t+\frac{i\gamma}{2})^2} \, dt = \lim\limits_{R,S \to \infty} \int\limits_{-R}^{S} e^{-t^2} \, dt = \sqrt{\pi}.
\]
Consequently, (2.1.5), (2.1.4), and (2.1.3) combine to show that $\hat{G}(\gamma) = \sqrt{\pi} e^{-\frac{\gamma^2}{4}}$ for every real number $\gamma$.

The next approach to this calculation – taught to the writer by Professor Roger Smith – also involves function theory, albeit in a different manner. Define
\[
U_1(z) := \sqrt{\pi} e^{-\frac{z^2}{4}} \quad \text{and} \quad U_2(z) := \int\limits_{-\infty}^{\infty} e^{-t^2} e^{-it \bar{z}} \, dt, \quad z \in \mathbb{C}.
\]
Let $y$ be a real number. A simple change of variables in the integral in (2.1.3) reveals that
\[
U_2(iy) = \int_{-\infty}^{\infty} e^{-t^2} e^{iy} dt = e^{-\frac{y^2}{4}} \int_{-\infty}^{\infty} e^{-(t+\frac{y}{2})^2} dt = \sqrt{\pi} e^{\frac{y^2}{4}} = U_1(iy). \tag{2.1.7}
\]

The function $U_1$ is entire, and we now show that $U_2$ is also entire. Let $z$ and $h$ be complex numbers, and assume that $0 < |h| \leq 1$. We see that
\[
\frac{U_2(z + h) - U_2(z)}{h} = \int_{-\infty}^{\infty} e^{-t^2} e^{-itz} \left[ \frac{e^{-ith} - 1}{h} \right] dt. \tag{2.1.8}
\]

Let $z = a + ib$, so that $|e^{-itz}| = e^{tb} \leq e^{\vert b \vert}$ for every $t \in \mathbb{R}$. Now
\[
\left| \frac{e^{-ith} - 1}{h} \right| = \left| \sum_{n=1}^{\infty} \frac{(-it)^n}{n!} - h^{n-1} \right| \leq e^{\vert t \vert}, \quad t \in \mathbb{R},
\]
the function $t \mapsto e^{-t^2} e^{\vert t \vert(1+ib)}$ is integrable on $(-\infty, \infty)$, and
\[
\lim_{h \to 0} \frac{e^{-ith} - 1}{h} = -it \quad \text{for every real number } t,
\]
and the Dominated Convergence Theorem may be applied to (2.1.8) to obtain
\[
U_2'(z) = \lim_{h \to 0} \frac{U_2(z + h) - U_2(z)}{h} = \int_{-\infty}^{\infty} e^{-t^2} e^{-it(-it)} dt.
\]

Thus $U_1$ and $U_2$ are entire functions which coincide on the imaginary axis (vide (2.1.7)), so they must coincide throughout the complex plane. In particular,
\[
\sqrt{\pi} e^{-\frac{y^2}{4}} = U_1(\gamma) = U_2(\gamma) = \int_{-\infty}^{\infty} e^{-t^2} e^{-it\gamma} dt = \hat{G}(\gamma), \quad \gamma \in \mathbb{R}.
\]

Our third and final method is by far the most elementary; it involves an elegant application of both parts of Theorem 2.1.5. In the calculations below, the second part of that theorem is used in the first step, whilst the first part is utilized in the penultimate step:
\[
(\hat{G}'(\gamma)) = \int_{-\infty}^{\infty} (-it)e^{-t^2} e^{-it\gamma} dt = \frac{i}{2} \int_{-\infty}^{\infty} G'(t)e^{-it\gamma} dt = \frac{i}{2} \left[ i\gamma \hat{G}(\gamma) \right] = -\frac{\gamma}{2} \hat{G}(\gamma), \quad \gamma \in \mathbb{R}.
\]

Basic theory of ordinary differential equations teaches us that the solution to the linear equation above is given by $\hat{G}(\gamma) = \hat{G}(0)e^{-\frac{\gamma^2}{4}} = \sqrt{\pi} e^{-\frac{\gamma^2}{4}}, \quad \gamma \in \mathbb{R}$, the final equation stemming from (2.1.3).

§2.2. Convolutions and approximate identities

Here we continue our study of the basic theory of Fourier transforms. In particular, we discuss the notion of convolution and some of its basic properties. The reader will notice considerable similarity between what follows here and the corresponding ideas discussed in the context of Fourier series.
**Definition 2.2.1.** Suppose that \( f \) and \( g \) belong to \( L^1(\mathbb{R}) \). The 
convolution of \( f \) and \( g \) is defined as follows:
\[
(f * g)(x) := \int_{-\infty}^{\infty} f(x-t)g(t) \, dt,
\]
whenever the integral exists.

**Theorem 2.2.2.** Let \( f, g \in L^1(\mathbb{R}) \). Then for almost all \( x \in \mathbb{R} \), the function \( t \mapsto f(x-t)g(t) \) belongs to \( L^1(\mathbb{R}) \). In particular, \( (f * g)(x) \) is well-defined for all such \( x \). Moreover, \( f * g \) also belongs to \( L^1(\mathbb{R}) \), and \( \|f * g\|_{L^1(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})} \).

**Proof.** See, for example, [HS, page 396].

**Theorem 2.2.3.** If \( f \) and \( g \) belong to \( L^1(\mathbb{R}) \), then \( \hat{f} \ast \hat{g} = \hat{f} \hat{g} \) for every real number \( \gamma \).

**Proof.** EXERCISE.

**Theorem 2.2.4.** Suppose that \( f, g, h \in L^1(\mathbb{R}) \). The following hold:

(i) \( f \ast (g * h) = (f \ast g) \ast h \).

(ii) \( f \ast (g + h) = f \ast g + f \ast h \).

(iii) If \( \alpha \) is any scalar, then \( (\alpha f \ast g) \ast h = \alpha (f \ast h) + (g \ast h) \).

**Proof.** EXERCISE.

**Remark 2.2.5.** One concludes from the foregoing results that \( L^1(\mathbb{R}) \) is an algebra with respect to the convolution operation. That this algebra does not possess an identity element (unit) is seen as follows: If there is such a unit, say \( E \), then \( E \ast f = f \) for every \( f \in L^1(\mathbb{R}) \), so Theorem 2.2.3 asserts that \( \hat{E} \hat{f}(\gamma) = \hat{f}(\gamma) \) for every \( f \in L^1(\mathbb{R}) \) and every real number \( \gamma \). Choosing \( f(t) := e^{-t^2}, t \in \mathbb{R} \), we find, via Example 2.1.7, that \( \hat{E}(\gamma) [\sqrt{\pi} e^{-\gamma^2/4}] = \sqrt{\pi} e^{-\gamma^2/4} \) for every \( \gamma \in \mathbb{R} \), whence \( \hat{E}(\gamma) = 1 \) for every real number \( \gamma \). However, this is in direct violation of the Riemann-Lebesgue Lemma for Fourier Transforms (Theorem 2.1.3(iii)), proving that no such \( E \) can exist.

In view of the remark above, we seek the ‘next-best’ alternative to an identity. Specifically, we have the following definition.

**Definition 2.2.6.** A collection, \( \{k_\lambda : \lambda > 0\} \), of functions in \( L^1(\mathbb{R}) \) is called an approximate identity for \( L^1(\mathbb{R}) \) if \( \lim_{\lambda \to \infty} \|k_\lambda \ast f - f\|_{L^1(\mathbb{R})} = 0 \) for every \( f \in L^1(\mathbb{R}) \).

Our next task is to develop an effective way to construct approximate identities, and we begin with a simple, yet very useful lemma.

**Lemma 2.2.7.** Suppose that \( H \) is continuous throughout the real line. Assume further that
\( H \in L^1(\mathbb{R}), \) and that \( \int_{-\infty}^{\infty} H(t) \, dt = 1. \) Define \( H_\lambda(t) := \lambda H(\lambda t), t \in \mathbb{R}, \lambda > 0. \) The following hold:

(i) \( \int_{-\infty}^{\infty} H_\lambda(t) \, dt = 1 \) for every \( \lambda. \)

(ii) \( \int_{-\infty}^{\infty} |H_\lambda(t)| \, dt = \|H\|_{L^1(\mathbb{R})} \) for every \( \lambda. \)

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\[(iii) \lim_{\lambda \to \infty} \int_{|t| > \delta} |H_\lambda(t)| \, dt = 0 \text{ for every fixed positive number } \delta.\]

**Proof.** The first two assertions follow directly from a simple change of variables \((u = \lambda t)\), so we move on to the final clause. The same change of variables also shows that
\[
\int_{|t| > \delta} |H_\lambda(t)| \, dt = \int_{|u| > \lambda \delta} |H(u)| \, du, = o(1), \quad \lambda \to \infty,
\]
because \(H\) is absolutely integrable on \(\mathbb{R}\).

The preceding lemma leads to the following convenient recipe for constructing approximate identities for \(L^1(\mathbb{R})\).

**Theorem 2.2.8.** Suppose that \(H\) is a function satisfying all the hypotheses of Lemma 2.2.7. For each \(\lambda > 0\), define \(k_\lambda(t) := \lambda H(\lambda t)\), \(t \in \mathbb{R}\). Then the family \(\{k_\lambda : \lambda > 0\}\) is an approximate identity for \(L^1(\mathbb{R})\).

**Proof.** Suppose that \(f \in L^1(\mathbb{R})\), and that \(\epsilon > 0\) is given. Choose a positive number \(\delta\) such that
\[
\int_{-\infty}^{\infty} |f(x-t) - f(x)| \, dx < \frac{\epsilon}{2(1 + \|H\|_{L^1(\mathbb{R})})}, \quad \text{whenever } |t| \leq \delta.
\] (2.2.1)

That such a \(\delta\) exists is a well-known fact from analysis (which we have already used in the proof of Theorem 2.1.3(iii)); a proof of this fact may be found, for example, in [HS, page 199].* As
\[
(k_\lambda \ast f)(x) - f(x) = \int_{-\infty}^{\infty} k_\lambda(t)[f(x-t) - f(t)] \, dt,
\]
we see that
\[
\int_{-\infty}^{\infty} |(k_\lambda \ast f)(x) - f(x)| \, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_\lambda(t)[f(x-t) - f(t)] \, dt \, dx \leq \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} |k_\lambda(t)| |f(x-t) - f(x)| \, dt \right] \, dx.
\] (2.2.2)

Each of the functions \(k_\lambda\) and \(f\) belongs to \(L^1(\mathbb{R})\), so the iterated integral
\[
\int_{-\infty}^{\infty} |k_\lambda(t)| \left[ \int_{-\infty}^{\infty} |f(x-t) - f(x)| \, dx \right] \, dt
\]
is convergent, and Fubini’s Theorem permits an interchange in the order of integration in (2.2.2). This leads to the relations
\[
\|k_\lambda \ast f - f\|_{L^1(\mathbb{R})} \leq \int_{-\infty}^{\infty} |k_\lambda(t)| \left[ \int_{-\infty}^{\infty} |f(x-t) - f(x)| \, dx \right] \, dt
\]

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\[
\begin{align*}
&= \int_{-\delta}^{\delta} |k(\lambda)(t)| \left[ \int_{-\infty}^{\infty} |f(x-t) - f(x)| \, dx \right] \, dt \\
&\quad + \int_{|t|>\delta} |k(\lambda)(t)| \left[ \int_{-\infty}^{\infty} |f(x-t) - f(x)| \, dx \right] \, dt \\
&=: I_1 + I_2. \tag{2.2.3}
\end{align*}
\]

As \( \int_{-\delta}^{\delta} |k(\lambda)(t)| \, dt \leq \int_{-\infty}^{\infty} |k(\lambda)(t)| \, dt = \|H\|_{L^1(\mathbb{R})} \) (Lemma 2.2.7(ii)), we find from (2.2.1) and (2.2.3) that
\[
|I_1| < \frac{\epsilon}{2}. \tag{2.2.4}
\]

Now the third assertion in Lemma 2.2.7 provides a positive number \( \lambda_0 \) such that
\[
\int_{|t|>\delta} |k(\lambda)(t)| \, dt < \frac{\epsilon}{2(1 + \|f\|_{L^1(\mathbb{R})})}, \quad \lambda \geq \lambda_0, \tag{2.2.5}
\]
whence
\[
|I_2| = \int_{|t|>\delta} |k(\lambda)(t)| \left[ \int_{-\infty}^{\infty} |f(x-t) - f(x)| \, dx \right] \, dt \leq 2\|f\|_{L^1(\mathbb{R})} \int_{|t|>\delta} |k(\lambda)(t)| \, dt < \frac{\epsilon}{2}, \quad \lambda \geq \lambda_0. \tag{2.2.6}
\]

Combining (2.2.3) with (2.2.4) and (2.2.6), we obtain
\[
\|k(\lambda)f - f\|_{L^1(\mathbb{R})} < \epsilon, \quad \lambda \geq \lambda_0,
\]
and this finishes the proof.

We now discuss three important examples of approximate identities, each constructed according to the prescription given by the foregoing result.

**Example 2.2.9.** (i) Let
\[
p(t) := \frac{1}{\pi(1 + t^2)}, \quad t \in \mathbb{R}, \quad \text{and} \quad k(\lambda)(t) := \lambda p(\lambda t), \quad t \in \mathbb{R}, \quad \lambda > 0.
\]
The function \( p \) satisfies all the conditions of Lemma 2.2.7, hence \( \{k(\lambda) : \lambda > 0\} \) is an approximate identity for \( L^1(\mathbb{R}) \) (Theorem 2.2.8).

(ii) Let
\[
G(t) := \frac{e^{-t^2}}{\sqrt{\pi}}, \quad t \in \mathbb{R}, \quad \text{and} \quad k(\lambda)(t) := \lambda G(\lambda t), \quad t \in \mathbb{R}, \quad \lambda > 0.
\]
The function \( G \) satisfies all the hypotheses of Lemma 2.2.7 (*vide* (2.1.3) in particular), hence \( \{k(\lambda) : \lambda > 0\} \) is an approximate identity for \( L^1(\mathbb{R}) \).
(iii) Let
\[
\omega(t) := \begin{cases} 
\frac{1}{2\pi} \left[ \frac{\sin(t/2)}{t/2} \right]^2, & \text{if } t \in \mathbb{R} \setminus \{0\}; \\
1/2\pi, & \text{if } t = 0,
\end{cases}
\]
and \(k_\lambda(t) := \lambda \omega(\lambda t), t \in \mathbb{R}, \lambda > 0\). We shall show that \(\omega\) also satisfies the conditions set forth in Lemma 2.2.7. As \(\omega\) is a nonnegative function which is continuous throughout the real line, all that needs to be shown is that
\[
\int_{-\infty}^{\infty} \omega(t) \, dt = 1.
\]
We prove this using some facts from the theory of Fourier series.

Recall from Theorem 1.2.7(iii) that the Fejér kernel (of degree \(N\)) is given by
\[
K_N(t) = \frac{1}{N+1} \left[ \frac{\sin^2((N+1)t)}{\sin^2(t/2)} \right], \quad |t| \leq \pi,
\]
with the understanding that \(K_N(0) = N+1 = \lim_{t \to 0} K_N(t)\). Furthermore, we also know (Theorem 1.2.8) that
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(t) \, dt = 1,
\]
and that
\[
\lim_{N \to \infty} \frac{1}{2\pi} \int_{|t| \geq \delta} K_N(t) \, dt = 0,
\]
for every fixed \(\delta \in (0, \pi)\). These two facts combine to show that
\[
\lim_{N \to \infty} \frac{1}{2\pi} \int_{-\delta}^{\delta} K_N(t) \, dt = 1,
\]
for every fixed \(\delta \in (0, \pi)\).

Define
\[
F_N(t) := (N+1)\omega((N+1)t) = \begin{cases} 
\frac{1}{2\pi(N+1)} \left[ \frac{\sin((N+1)t/2)}{(N+1)t/2} \right]^2, & \text{if } t \in \mathbb{R} \setminus \{0\}; \\
(N+1)/2\pi, & \text{if } t = 0.
\end{cases}
\]
Suppose that \(0 < \delta \leq \pi\) is fixed (but arbitrary). As the function \(t \mapsto \frac{\sin t}{t}\) is an even function which is nonnegative, continuous, and decreasing in the interval \([0, \pi]\) (as usual the function in question is defined to be 1 at \(t = 0\)), we find that
\[
\frac{F_N(t)}{K_N(t)} = \frac{\omega(t)}{K_N(t)} \geq \omega(\delta), \quad |t| \leq \delta,
\]
or, equivalently, that
\[
\omega(\delta) K_N(t) \leq F_N(t), \quad |t| \leq \delta.
\]
On the other hand, the inequality \(|\sin \theta| \leq |\theta|, \theta \in \mathbb{R}\), implies that
\[
F_N(t) \leq \frac{1}{2\pi} K_N(t), \quad |t| \leq \pi.
\]
From (2.2.9) and (2.2.10) we infer that
\[
\omega(\delta) \int_{-\delta}^{\delta} K_N(t) \, dt \leq \int_{-\delta}^{\delta} F_N(t) \, dt \leq \frac{1}{2\pi} \int_{-\delta}^{\delta} K_N(t) \, dt.
\]
Now the Monotone Convergence Theorem (used in the last step below) ensures that
\[
\lim_{N \to \infty} \int_{-\delta}^{\delta} F_N(t) \, dt = \lim_{N \to \infty} \int_{-\delta}^{\delta} (N+1)\omega((N+1)t) \, dt = \lim_{N \to \infty} \int_{-(N+1)\delta}^{(N+1)\delta} \omega(t) \, dt = \int_{-\infty}^{\infty} \omega(t) \, dt,
\]
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and using this and (2.2.7) in (2.2.11) leads to the relations

\[ 2\pi \omega(\delta) \leq \int_{-\infty}^{\infty} \omega(t) \, dt \leq 1. \]  

(2.2.12)

This being true for every \( \delta \in (0, \pi) \), we may let \( \delta \) approach zero in (2.2.12) to conclude, via the fact that \( \lim_{\delta \to 0^+} 2\pi \omega(\delta) = 1 \), that \( \int_{-\infty}^{\infty} \omega(t) \, dt = 1 \).

\section*{2.3. Inversion of the Fourier transform}

This section addresses the issue of ‘inverting’ the Fourier transform, namely, that of recovering the original function from its Fourier transform. We intend to present two general theorems in this direction.

The next five results will provide the background for the first of the main inversion theorems.

\textbf{Lemma 2.3.1.} Suppose that \( a \) and \( b \) are fixed real numbers, with \( a < b \). Let \( \alpha \) be a bounded nondecreasing function on \([a,b]\), and let \( h \) be continuous on \([a,b]\). Then there exists a number \( \xi \in [a,b] \) such that

\[ \int_{a}^{b} \alpha(x) h(x) \, dx = \alpha(a^+) \int_{a}^{\xi} h(x) \, dx + \alpha(b^-) \int_{\xi}^{b} h(x) \, dx. \]

\textbf{Proof.} Define

\[ f(x) := \begin{cases} 
\alpha(a^+), & \text{if } x = a; \\
\alpha(x), & \text{if } a < x < b; \\
\alpha(b^-), & \text{if } x = b,
\end{cases} \]

and apply (a version of) the Second Mean-Value Theorem for Integrals (see [B, page 233]).

\textbf{Lemma 2.3.2.}

\[ \int_{0}^{\infty} \frac{\sin(t)}{t} \, dt := \lim_{\epsilon \to 0^+} \int_{\epsilon}^{R} \frac{\sin t}{t} \, dt = \frac{\pi}{2}. \]

\textbf{Proof.} Let \( 0 < \epsilon < R \) and consider \( I(\epsilon, R) := \int_{\epsilon}^{R} \frac{\sin^2 x}{x^2} \, dx \). Integration by parts shows that

\[ I(\epsilon, R) = \frac{\sin^2 \epsilon}{\epsilon} - \frac{\sin^2 R}{R} + \int_{\epsilon}^{R} \frac{2 \sin x \cos x}{x} \, dx \]

\[ = \frac{\sin^2 \epsilon}{\epsilon} - \frac{\sin^2 R}{R} + \int_{\epsilon}^{R} \frac{\sin(2x)}{x} \, dx \]

\[ = \frac{\sin^2 \epsilon}{\epsilon} - \frac{\sin^2 R}{R} + \int_{2\epsilon}^{2R} \frac{\sin(t)}{t} \, dt. \]  

(2.3.1)

On the other hand, recalling the function $\omega$ from Example 2.2.9(iii), we also find that

$$I(\epsilon, R) = 2\pi \int_{\epsilon}^{R} \omega(2x) \, dx = \pi \left[ \int_{-\epsilon}^{\epsilon} \omega(2x) \, dx + \int_{\epsilon}^{R} \omega(2x) \, dx \right]$$

$$= \frac{\pi}{2} \left[ \int_{-2\epsilon}^{-2\epsilon} \omega(t) \, dt + \int_{2\epsilon}^{2\epsilon} \omega(t) \, dt \right], \quad (2.3.2)$$

the penultimate step coming from the fact that $\omega$ is an even function. As

$$\lim_{\epsilon \to 0^+} \frac{\sin^{2} \epsilon}{\epsilon} = 0 = \lim_{R \to \infty} \frac{\sin^{2} R}{R},$$

and Example 2.2.9(iii) demonstrated that

$$\lim_{\epsilon \to 0^+, R \to \infty} \left[ \int_{-2\epsilon}^{-2\epsilon} \omega(t) \, dt + \int_{2\epsilon}^{2\epsilon} \omega(t) \, dt \right] = 1,$$

we see from (2.3.1) and (2.3.2) that

$$\lim_{\epsilon \to 0^+, R \to \infty} R \int_{\epsilon}^{\epsilon} \frac{\sin t}{t} \, dt = \frac{\pi}{2}.$$  

\[\blacksquare\]

**Corollary 2.3.3.** Let

$$\text{sinc}(t) := \begin{cases} \frac{\sin t}{t}, & \text{if } t \neq 0; \\ 1, & \text{if } t = 0. \end{cases}$$

There exists a positive constant $\Delta$ such that $\left| \int_{a}^{b} \text{sinc}(t) \, dt \right| \leq \Delta$ for every pair of real numbers $a$ and $b$ such that $0 \leq a < b$.

**Proof.** Lemma 2.3.2 provides a positive number $R_0$ such that $\left| \int_{0}^{R} \text{sinc}(t) \, dt - \frac{\pi}{2} \right| < 1$ for every $R \geq R_0$. Therefore,

$$\left| \int_{0}^{R} \text{sinc}(t) \, dt \right| = \left| \int_{0}^{R} \text{sinc}(t) \, dt - \frac{\pi}{2} + \frac{\pi}{2} \right| < 1 + \frac{\pi}{2}, \quad R \geq R_0. \quad (2.3.3)$$

Suppose now that $0 \leq a < b$. If $b \leq R_0$, then the inequality $|\text{sinc}(t)| \leq 1$, $t \in \mathbb{R}$, provides the bound

$$\left| \int_{a}^{b} \text{sinc}(t) \, dt \right| \leq \int_{0}^{R_0} |\text{sinc}(t)| \, dt \leq R_0. \quad (2.3.4)$$
If \( a \leq R_0 < b \), then (2.3.3) and (2.3.4) lead to the relations

\[
\left| \int_a^b \text{sinc}(t) \, dt \right| = \left| \int_0^b \text{sinc}(t) \, dt - \int_0^a \text{sinc}(t) \, dt \right| \leq 1 + \frac{\pi}{2} + R_0. 
\]  
(2.3.5)

Finally, if \( R_0 < a < b \), then

\[
\left| \int_a^b \text{sinc}(t) \, dt \right| = \left| \int_0^b \text{sinc}(t) \, dt - \int_0^a \text{sinc}(t) \, dt \right| \leq 2 \left(1 + \frac{\pi}{2}\right), 
\]  
(2.3.6)

via (2.3.3). The required result follows from (2.3.4)–(2.3.6).

\[ \text{Lemma 2.3.4.} \] Suppose that \( \alpha \) is a bounded nondecreasing function in the interval \([0, \delta]\) for some \( \delta > 0 \). Then

\[
\lim_{R \to \infty} \frac{1}{\pi} \int_0^\delta \alpha(x) \frac{\sin(Rx)}{x} \, dx = \frac{\alpha(0^+)}{2}.
\]

**Proof.** Let \( \Delta \) be the constant from the preceding lemma, and suppose that \( \alpha(0^+) = 0 \). Let \( \epsilon > 0 \) be given, and choose a positive number \( \eta < \delta \) such that

\[
|\alpha(x)| \leq \frac{\epsilon}{\Delta} \quad \text{whenever} \quad 0 < x \leq \eta.
\]

As \( \alpha \) is nondecreasing, it follows, in particular, that

\[
|\alpha(\eta^-)| \leq \frac{\epsilon}{2(1 + \Delta)}.
\]  
(2.3.7)

Let \( R \) be any positive number. Lemma 2.3.1 guarantees a number \( \xi \in [0, \eta] \) such that

\[
\int_0^\eta \alpha(x) \frac{\sin(Rx)}{x} \, dx = \alpha(\eta^-) \int_\xi^\eta \frac{\sin(Rx)}{x} \, dx = \alpha(\eta^-) \int_{R\xi}^{R\eta} \frac{\sin t}{t} \, dt,
\]

whence Lemma 2.3.3 and (2.3.7) provide the estimate

\[
\left| \int_0^\eta \alpha(x) \frac{\sin(Rx)}{x} \, dx \right| = \left| \alpha(\eta^-) \int_{R\xi}^{R\eta} \frac{\sin t}{t} \, dt \right| \leq \frac{\epsilon \Delta}{2(1 + \Delta)} \leq \frac{\epsilon}{2}.
\]  
(2.3.8)

As the function \( x \mapsto \frac{\alpha(x)}{x} \) is integrable on the interval \([\eta, \delta]\), the Riemann–Lebesgue Lemma for Fourier Transforms (applied to the function \( x \mapsto \frac{\alpha(x)}{x} \chi_{[\eta, \delta]}(x) \)) ensures that

\[
\lim_{R \to \infty} \int_{-\infty}^\infty \frac{\alpha(x)}{x} \chi_{[\eta, \delta]}(x) \sin(Rx) \, dx = \lim_{R \to \infty} \int_{\eta}^\delta \frac{\alpha(x)}{x} \sin(Rx) \, dx = 0.
\]

Consequently, there is a positive number \( R_1 \) such that

\[
\left| \int_{\eta}^{\delta} \frac{\alpha(x)}{x} \sin(Rx) \, dx \right| < \frac{\epsilon}{2}, \quad R \geq R_1.
\]  
(2.3.9)
Combining (2.3.8) and (2.3.9), we see that

$$\left| \int_{0}^{\delta} \alpha(x) \frac{\sin(Rx)}{x} \, dx \right| = \int_{0}^{\eta} \alpha(x) \frac{\sin(Rx)}{x} \, dx + \int_{\eta}^{\delta} \alpha(x) \frac{\sin(Rx)}{x} \, dx < \epsilon, \quad R \geq R_1.$$ 

Thus \( \lim_{R \to \infty} \int_{0}^{\delta} \alpha(x) \frac{\sin(Rx)}{x} \, dx = 0 \), and the stated result obtains under the assumption that \( \alpha(0^+) = 0 \).

Suppose now that \( \alpha(0^+) \neq 0 \), and define \( \beta(x) := \alpha(x) - \alpha(0^+) \). Then \( \beta \) is a bounded nondecreasing function on \([0, \delta]\), and \( \beta(0^+) = 0 \). So the foregoing analysis shows that

$$0 = \lim_{R \to \infty} \frac{1}{2\pi} \int_{0}^{\delta} \beta(x) \frac{\sin(Rx)}{x} \, dx = \lim_{R \to \infty} \frac{1}{2\pi} \int_{0}^{\delta} \left[ \alpha(x) - \alpha(0^+) \right] \frac{\sin(Rx)}{x} \, dx. \quad (2.3.10)$$

Now Lemma 2.3.2 implies that

$$\lim_{R \to \infty} \frac{\alpha(0^+)}{\pi} \int_{0}^{\delta} \frac{\sin(Rx)}{x} \, dx = \lim_{R \to \infty} \frac{\alpha(0^+)}{\pi} \int_{0}^{R \delta} \frac{\sin t}{t} \, dt = \frac{\alpha(0^+)}{2},$$

and using this in (2.3.10) confirms that \( \lim_{R \to \infty} \frac{1}{2\pi} \int_{0}^{\delta} \alpha(x) \frac{\sin(Rx)}{x} \, dx = \frac{\alpha(0^+)}{2} \).

**Corollary 2.3.5.** Suppose that \( \lambda \) is a function of bounded variation in the interval \([0, \delta]\), for some \( \delta > 0 \). Then

$$\lim_{R \to \infty} \frac{1}{2\pi} \int_{0}^{\delta} \lambda(x) \frac{\sin(Rx)}{x} \, dx = \frac{\lambda(0^+)}{2}.$$ 

**Proof.** Write \( \lambda \) as a difference of two bounded, nondecreasing functions, and apply Lemma 2.3.4. \( \square \)

We are now ready for the first inversion theorem.

**Theorem 2.3.6.** (Jordan’s Inversion Theorem) Let \( f \in L^1(\mathbb{R}) \), and assume that \( f \) is of bounded variation in the interval \([u - \delta, u + \delta]\), for some real number \( u \), and some positive number \( \delta \). Then

$$\lim_{R \to \infty} \frac{1}{2\pi} \int_{-R}^{R} \hat{f}(\gamma)e^{iux} \, d\gamma = \frac{f(u^+) + f(u^-)}{2}.$$ 

**Proof.** Let \( R > 0 \), and define

$$S_R(u) := \frac{1}{2\pi} \int_{-R}^{R} \hat{f}(\gamma)e^{iux} \, d\gamma = \frac{1}{2\pi} \int_{-R}^{R} e^{iuy} \left[ \int_{-\infty}^{\infty} f(x)e^{-i\gamma x} \, dx \right] \, d\gamma.$$
As \( f \in L^1(\mathbb{R}) \), the iterated integral above is absolutely convergent, so Fubini’s Theorem allows us to write

\[
S_R(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \left[ \int_{-R}^{R} e^{-i\gamma(x-u)} d\gamma \right] dx = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \frac{\sin(R(u-x))}{u-x} dx \\
= \frac{1}{\pi} \int_{-\infty}^{\infty} f(u-x) x \sin(Rx) dx.
\]  

(2.3.11)

The function \( x \mapsto \frac{\sin(Rx)}{x} \) being even, we find that

\[
S_R(u) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u-x) x \sin(Rx) dx = \frac{1}{\pi} \int_{-\infty}^{0} f(u-x) x \sin(Rx) dx + \frac{1}{\pi} \int_{0}^{\infty} f(u-x) x \sin(Rx) dx \\
= \frac{1}{\pi} \int_{0}^{\infty} [f(u+x) + f(u-x)] x \sin(Rx) dx.
\]  

(2.3.12)

Define \( \lambda(x) := f(u+x) + f(u-x) \); our hypothesis ensures that \( \lambda \) is of bounded variation in \([0, \delta]\).

Rewriting (2.3.12) as

\[
S_R(u) = \frac{1}{\pi} \int_{0}^{\delta} \lambda(x) x \sin(Rx) dx + \frac{1}{\pi} \int_{\delta}^{\infty} \lambda(x) x \sin(Rx) dx =: A_R(u) + B_R(u),
\]  

(2.3.13)

we find that

\[
\lim_{R \to \infty} A_R(u) = \lambda(0^+) \frac{1}{2} = \frac{f(u^+) + f(u^-)}{2},
\]  

(2.3.14)

via Corollary 2.3.5. On the other hand, as \( f \in L^1(\mathbb{R}) \), the function \( x \mapsto \frac{\lambda(x)}{x} \chi_{(\delta, \infty)}(x) = \frac{f(u+x) + f(u-x)}{x} \chi_{(\delta, \infty)}(x) \) is integrable on \((\infty, \infty)\), so the Riemann-Lebesgue Lemma for Fourier Transforms implies that

\[
\lim_{R \to \infty} B_R(u) = 0.
\]  

(2.3.15)

Combining (2.3.14) and (2.3.15) with (2.3.13) yields the required result.

A simple consequence of Jordan’s Theorem is the following:

**Corollary 2.3.7.** Suppose that \( f \in L^1(\mathbb{R}) \), and that it is of bounded variation in an interval \([u-\delta, u+\delta]\), for some \( u \in \mathbb{R} \), and some \( \delta > 0 \). If \( f \) is continuous at \( u \), then

\[
\lim_{R \to \infty} \frac{1}{2\pi} \int_{-R}^{R} \hat{f}(\gamma) e^{iu\gamma} d\gamma = f(u).
\]

**Proof.** This is evident.
The next inversion theorem we shall consider will be of a different character, and very similar in spirit to Theorem 1.3.5 on Fourier series. We begin with a brief discussion of the notion of \((C, 1)\) summability for integrals.

**Definition 2.3.8.** Suppose that \(a\) is an integrable function on the interval \([-R, R]\) for every positive number \(R\). We say that the integral \(\int_{-\infty}^{\infty} a(x) \, dx\) is \((C, 1)\) summable to the value \(A\), and denote this by \(\int_{-\infty}^{\infty} a(x) \, dx = A\) if \(\lim_{R \to \infty} \int_{-R}^{R} \left(1 - \frac{|x|}{R}\right) a(x) \, dx = A\).

**Remark 2.3.9.** The reader should note that Definition 2.3.8 is the integral analogue of the notion of \((C, 1)\) summability for series. Indeed, an infinite series \(\sum_{k=-\infty}^{\infty} a_k\), with partial sums \(s_j := \sum_{k=-j}^{j} a_k\), is said to be \((C, 1)\) summable to the value \(A\), provided that \(\lim_{n \to \infty} \frac{1}{n+1} \sum_{j=0}^{n} s_j = \lim_{n \to \infty} \frac{1}{n+1} \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) a_k = A\).

In view of what obtains in the theory of infinite series, the following should come as no surprise.

**Proposition 2.3.10.** If \(a \in L^1(\mathbb{R})\), and \(\int_{-\infty}^{\infty} a(x) \, dx = A\), then \(\int_{-\infty}^{\infty} a(x) \, dx = A\) if \(\int_{-\infty}^{\infty} a(x) \, dx = A\) if \((C, 1)\) summable.

**Proof.** Suppose that \(R > 0\). As \(a \in L^1(\mathbb{R})\), it is integrable on \([-R, R]\). Define \(a_R(x) := \left\{ \begin{array}{ll} \left(1 - \frac{|x|}{R}\right) a(x), & \text{if } |x| \leq R; \\ 0, & \text{otherwise}. \end{array} \right.\)

Then \(|a_R(x)| \leq |a(x)|\) and \(\lim_{R \to \infty} a_R(x) = a(x)\) for almost every real number \(x\). So the integrability of \(a\) (on \(\mathbb{R}\)), coupled with the Dominated Convergence Theorem, implies that \(\lim_{R \to \infty} \int_{-R}^{R} \left(1 - \frac{|x|}{R}\right) a(x) \, dx = \lim_{R \to \infty} \int_{-\infty}^{\infty} a_R(x) \, dx = \int_{-\infty}^{\infty} a(x) \, dx = A\);

in other words, \(\int_{-\infty}^{\infty} a(x) \, dx = A\) if \(\int_{-\infty}^{\infty} a(x) \, dx = A\) if \((C, 1)\) summable. \(\blacksquare\)

Before moving to the next major theorem of the section, we assemble some preliminary material, beginning with some basic facts from general analysis, as found, for example, in [HS, pp. 277-278].

Given \(f \in L^1(\mathbb{R})\), a real number \(u\) is said to be a **Lebesgue point** of \(f\) if

\[
\lim_{h \to 0^+} \frac{1}{h} \int_{0}^{h} \left| f(u + \tau) + f(u - \tau) - 2f(u) \right| \, d\tau = 0. \tag{2.3.16}
\]

The set of Lebesgue points of a function $f$ is called the Lebesgue set of $f$, and one of Lebesgue’s famous theorems asserts that, for any given $f \in L^1(\mathbb{R})$, almost every real number belongs to its Lebesgue set. The defining relation (2.3.16) shows that, if $f$ is continuous at $u$, then $u$ must be a Lebesgue point of $f$.

In what follows, the function $\omega$ from Example 2.2.9(iii) will play a crucial rôle. Recall that it is an even function which is continuous throughout the real line. If $\omega_R(t) := R \omega(Rt)$, $R > 0, \ t \in \mathbb{R}$, then the evenness of $\omega_R$, Example 2.2.9(iii), and Lemma 2.2.7(i) provide the relations

$$1 = \int_{-\infty}^{\infty} \omega(t) \, dt = \int_{-\infty}^{\infty} \omega_R(t) \, dt = 2 \int_{0}^{\infty} \omega_R(t) \, dt.$$  (2.3.17)

Our second inversion theorem for Fourier transforms may now be presented. The reader should note the similarity between its proof and that of (Lebesgue’s) Theorem 1.3.5 on Fourier series.

**Theorem 2.3.11.** If $f \in L^1(\mathbb{R})$, and $u$ is a Lebesgue point of $f$, then

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\gamma)e^{iu\gamma} \, d\gamma = f(u) \quad (C,1);$$

more precisely,

$$\lim_{R \to \infty} \frac{1}{2\pi} \int_{-R}^{R} \left(1 - \frac{|\gamma|}{R}\right) \hat{f}(\gamma)e^{iu\gamma} \, d\gamma = f(u).$$  (2.3.18)

In particular, (2.3.18) holds for almost every real number $u$, including every $u$ at which $f$ is continuous.

**Proof.** Let $R > 0$, and let $u$ be any real number. Defining

$$\Sigma_R(u) := \frac{1}{2\pi} \int_{-R}^{R} \left(1 - \frac{|\gamma|}{R}\right) \hat{f}(\gamma)e^{iu\gamma} \, d\gamma = \frac{1}{2\pi} \int_{-R}^{R} \left(1 - \frac{|\gamma|}{R}\right) e^{iu\gamma} \left[ \int_{-\infty}^{\infty} f(t)e^{-i\gamma t} \, dt \right] d\gamma,$$  (2.3.19)

our goal is to show that $\lim_{R \to \infty} \Sigma_R(u) = f(u)$ for every $u$ belonging to the Lebesgue set of $f$.

As $f \in L^1(\mathbb{R})$, the iterated integral in (2.3.19) converges absolutely, so Fubini’s Theorem and a straightforward calculation involving integration by parts reveal that

$$\Sigma_R(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \left[ \int_{-R}^{R} \left(1 - \frac{|\gamma|}{R}\right) e^{i\gamma(u-t)} \, d\gamma \right] dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \left[ \frac{\sin^2 \left( \frac{R(u-t)}{2} \right)}{\left( \frac{R(u-t)}{2} \right)^2} \right] dt.$$  (2.3.20)

Recalling the definitions of the functions $\omega$ and $\omega_R$, we see that (2.3.20) may be written as follows:

$$\Sigma_R(u) = \int_{-\infty}^{\infty} f(t)\omega_R(u-t) \, dt = \int_{-\infty}^{\infty} f(u-t)\omega_R(t) \, dt$$
\[
\begin{align*}
&= \int_{-\infty}^{0} f(u-t)\omega_R(t)\,dt + \int_{0}^{\infty} f(u-t)\omega_R(t)\,dt \\
&= \int_{0}^{\infty} f(u+t)\omega_R(-t)\,dt + \int_{0}^{\infty} f(u-t)\omega_R(t)\,dt \\
&= \int_{0}^{\infty} [f(u+t) + f(u-t)]\omega_R(t)\,dt,
\end{align*}
\]

\begin{equation}
(2.3.21)
\end{equation}

the final step being consequent upon the fact that \(\omega_R\) is an even function. Combining (2.3.21) with (2.3.17), one arrives at the equation

\[
\Sigma_R(u) - f(u) = \int_{0}^{\infty} [f(u+t) + f(u-t) - 2f(u)]\omega_R(t)\,dt.
\]

\begin{equation}
(2.3.22)
\end{equation}

Let \(u\) be a (fixed but arbitrary) Lebesgue point of \(f\). Define

\[
\begin{align*}
\phi(\tau) := |f(u+\tau) + f(u-\tau) - 2f(u)| & \quad \text{and} \quad \Phi(t) := \int_{0}^{t} \phi(\tau)\,d\tau.
\end{align*}
\]

As \(f \in L^1(\mathbb{R})\), \(\Phi\) is absolutely continuous on every closed and bounded subinterval of the real line, and \(\Phi' = \phi\) on every such interval. Let \(\epsilon > 0\) be given, and define \(\sigma := (2\pi\epsilon)/13\). Use (2.3.16) to choose an \(\eta > 0\) such that

\[
\Phi(t) \leq t\sigma, \quad 0 \leq t \leq \eta,
\]

\begin{equation}
(2.3.23)
\end{equation}

and decouple (2.3.22) as follows:

\[
\Sigma_R(u) - f(u) = \int_{0}^{\eta} [f(u+t) + f(u-t) - 2f(u)]\omega_R(t)\,dt + \int_{\eta}^{\infty} [f(u+t) + f(u-t) - 2f(u)]\omega_R(t)\,dt =: A\Sigma_R(u) + B\Sigma_R(u).
\]

\begin{equation}
(2.3.24)
\end{equation}

Consider \(B\Sigma_R(u)\). As \(0 \leq \sin^2(\theta) \leq 1\) for every real number \(\theta\), we see that

\[
\omega_R(t) = \frac{R}{2\pi} \left[ \frac{\sin^2 \left( \frac{Rt}{2} \right)}{\left( \frac{Rt}{2} \right)^2} \right] \leq \frac{2}{\pi R t^2}, \quad t \neq 0.
\]

\begin{equation}
(2.3.25)
\end{equation}

Therefore,

\[
|B\Sigma_R(u)| \leq \frac{2}{\pi R} \int_{\eta}^{\infty} \frac{|f(u+t) + f(u-t) - 2f(u)|}{t^2}\,dt \leq \frac{2}{\pi R \eta^2} \int_{\eta}^{\infty} |f(u+t) + f(u-t)|\,dt + \frac{4|f(u)|}{\pi R} \int_{\eta}^{\infty} \frac{dt}{t^2} \leq \frac{4\|f\|_{L^1(\mathbb{R})}}{\pi R \eta^2} + \frac{4|f(u)|}{\pi R \eta}.
\]
showing that
\[
\lim_{R \to \infty} B_R(u) = 0. \tag{2.3.26}
\]

Turning to \( A_R(u) \), choose, at the outset, a positive number \( R_0 \) such that \( 1/R < \eta \) for every \( R \geq R_0 \). Assume hereafter that \( R > R_0 \), and write
\[
A_R(u) = \frac{1}{R} \int_0^{1/R} [f(u + t) + f(u - t) - 2f(u)] \omega_R(t) \, dt + \frac{R}{2\pi} \Phi \left( \frac{1}{R} \right)
\]
\[
=: C_R(u) + D_R(u). \tag{2.3.27}
\]

As \(|\sin \theta| \leq |\theta|\) for every real number \( \theta \), we see that \( 0 \leq \omega(t) \leq \frac{1}{2\pi} \) for every \( t \in \mathbb{R} \), so \( 0 \leq \omega_R(t) \leq \frac{R}{2\pi} \) for every real number \( t \). Therefore,
\[
|C_R(u)| \leq \int_0^{1/R} |f(u + t) + f(u - t) - 2f(u)| \, dt = \frac{R}{2\pi} \Phi \left( \frac{1}{R} \right) \leq \frac{\sigma}{2\pi}, \tag{2.3.28}
\]
by (2.3.23) and the choice of \( R \).

On the other hand, \( 2.3.25 \) leads to the bound
\[
|D_R(u)| \leq 2\pi R \frac{\Phi(\eta)}{\eta^2} + 4\pi R \eta \int_0^{1/R} \Phi(t) \frac{t}{t^2} \, dt. \tag{2.3.29}
\]

Integrating the last integral in (2.3.29) by parts, and remembering that \( \Phi \) is a nonnegative function, we obtain
\[
\frac{2}{\pi R} \int_0^{\eta} \frac{\Phi(t)}{t^2} \, dt = \frac{2}{\pi R} \left[ \frac{\Phi(\eta)}{\eta^2} - R^2 \Phi \left( \frac{1}{R} \right) + \int_0^{\eta} \frac{\Phi(t)}{t^3} \, dt \right]
\]
\[
\leq \frac{2}{\pi R} \left[ \frac{\Phi(\eta)}{\eta^2} + \int_0^{\eta} \frac{\Phi(t)}{t^3} \, dt \right]. \tag{2.3.30}
\]

Recalling (2.3.23) and that \( R\eta > 1 \), we see that
\[
\frac{2}{\pi R} \left[ \frac{\Phi(\eta)}{\eta^2} + \int_0^{\eta} \frac{\Phi(t)}{t^3} \, dt \right]
\]
\[
\leq \frac{2\sigma}{\pi} + 4\pi R \int_0^{\eta} \frac{t}{t^2} \, dt
\]
\[
= \frac{2\sigma}{\pi} + 4\pi \left[ R - \frac{1}{\eta} \right] \leq \frac{2\sigma}{\pi} + \frac{4\sigma}{\pi}. \tag{2.3.31}
\]
From (2.3.29)–(2.3.31), we get the estimate

\[ |D_R(u)| \leq \frac{6\sigma}{\pi}, \quad R \geq R_0, \]

and combining this with (2.3.27) and (2.3.28) gives

\[ |A_R(u)| \leq \frac{13\sigma}{2\pi} = \epsilon, \quad R \geq R_0, \]

proving that

\[ \lim_{R \to \infty} A_R(u) = 0. \] (2.3.32)

The required result now follows from (2.3.24), (2.3.26), and (2.3.32).

We conclude the current chapter with two important consequences of Theorem 2.3.11.

**Corollary 2.3.12.** Suppose that \( f \) and \( \hat{f} \) belong to \( L^1(\mathbb{R}) \). If \( u \) is a Lebesgue point of \( f \), then

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\gamma) e^{iu\gamma} d\gamma = f(u). \] (2.3.33)

In particular, (2.3.33) holds for almost every real number \( u \), including every \( u \) at which \( f \) is continuous.

**Proof.** As \( \hat{\hat{f}} \in L^1(\mathbb{R}) \), the function \( \gamma \mapsto \hat{\hat{f}}(\gamma) e^{iu\gamma} \) is absolutely integrable on \( \mathbb{R} \), for every real number \( u \). Now combine Theorem 2.3.11 with Proposition 2.3.10.

**Corollary 2.3.13.** (Uniqueness Theorem for Fourier Transforms) Suppose that \( G, H \in L^1(\mathbb{R}) \), and that \( \hat{\hat{G}}(\gamma) = \hat{\hat{H}}(\gamma) \) for every \( \gamma \in \mathbb{R} \). Then \( G = H \) almost everywhere.

**Proof.** Define \( f := G - H \). Theorem 2.1.2(i) implies that \( \hat{\hat{f}}(\gamma) = \hat{\hat{G}}(\gamma) - \hat{\hat{H}}(\gamma) = 0 \) for every real number \( \gamma \); in particular, \( \hat{f} \in L^1(\mathbb{R}) \). Hence Corollary 2.3.12 proclaims that \( f = 0 \) almost everywhere, that is, \( G = H \) almost everywhere.