Summary

Volume by cross-sections

Let \( P_{x_0} = \{(x, y, z) : x = x_0\} \) be a plane perpendicular to the \( x \)-axis. Let \( S \) be a solid lying between \( P_a \) and \( P_b \) (here \( a < b \)) with these two planes just touching the solid. For \( a \leq x \leq b \), let \( A(x) \) represent the area of the cross-section obtained when the plane \( P_x \) intersects the solid \( S \). (Think of a knife cutting a loaf of bread into slices.) Provided that \( A(x) \) is integrable on \([a, b]\), the volume of the solid \( S \) is given by

\[
V = \int_a^b A(x) \, dx.
\]

Analogously, let \( S \) be a solid that lies between planes \( P_c \) and \( P_d \) that are perpendicular to the \( y \)-axis (here \( c < d \)) with these two planes just touching the solid. Then the volume of \( S \) is given by

\[
V = \int_c^d A(y) \, dy,
\]

where \( A(y) \) is the area of the cross-sectional intersection of \( S \) with a plane \( P_y \) parallel to \( P_c \) and \( P_d \), \( c \leq y \leq d \).

Procedure for computing volume

Keep in mind the sliced bread analogy!

1. The thickness of the bread is either \( dx \) or \( dy \) according to whether we slice the bread perpendicular to the \( x \)- or \( y \)-axis, respectively.

2. The area of a cross-sectional slice is \( A(x) \) or \( A(y) \).

3. The volume of a slice is \( dV = A(x) \, dx \) or \( dV = A(y) \, dy \).

4. Add ‘em all up and you get the volume of the loaf,

\[
V = \int \, dV = \int_a^b A(x) \, dx \quad \text{or} \quad V = \int \, dV = \int_c^d A(y) \, dy.
\]

But this isn’t necessarily your mother’s loaf of bread—unless your mom is a mathematician! The cross-sectional slices may have a variety of shapes: rectangular, square, triangular, circular, washer-shaped, etc. Not to worry: methodically follow the four-step procedure and you’ll have the volume of the loaf!

Geometrically depicting the slice

As you can imagine, Step 2—computing \( A(x) \) or \( A(y) \), the cross-sectional area of the slice—is the crux of the problem. While this takes some doing, it amounts to sketching and labeling a two-dimensional diagram. This is something we all can do with practice. In this lecture handout I’ll also draw a 3-D picture of the solid to help with visualization. Note, however, that this is not required when computing the volume of a solid.

Hand Examples

For specificity, we assume lengths are in centimeters.

438/2

Find the volume of the solid obtained by rotating the region bounded by \( y^2 = x^3 \), \( x = 4 \), and \( y = 0 \) about the \( x \)-axis.

Solution

Preliminary work  The planar region lies in the first octant. Its curved boundary is \( y = x^{3/2} \). Revolve region about \( x \)-axis.

Here is a sketch of a cross-section: a circular disk of radius \( y = x^{3/2} \), perpendicular to the \( x \)-axis with center on said axis. Finally, a 3-D picture of the solid is shown (with variable scaling).
The procedure for computing the volume of the solid follows.

1. The thickness of a cross-sectional slice is $dx$.
2. The area of this cross-section is
   \[ A(x) = \pi y^2 = \pi \left(\frac{x^3}{2}\right)^2 = \pi x^3. \]
3. The volume of the slice is $dV = A(x) \, dx = \pi x^3 \, dx$.
4. Now add up the volumes of the slices to obtain the volume of the entire solid.
   \[
   \int dV = \int_0^4 \pi x^3 \, dx = \left. \left(\frac{4}{4} \pi x^4\right) \right|_0^4 = (4^3 \pi) - (0) = 64\pi \approx 201.06
   \]
   The volume is $64\pi \approx 201$ cm$^3$.

438/11

Find the volume of the solid obtained by rotating the region bounded by the curves $y = x^4$ and $y = 1$ about the line $y = 2$.

Solution

Preliminary work  The planar region lies in the upper half-plane. Revolve the region about the line $y = 2$.

The cross-section is a circular washer (ring or annulus) of inner radius $r_i = 2 - 1 = 1$ and outer radius $r_o = 2 - x^4$, perpendicular to the line $y = 2$ with center on said line (its axis of symmetry). Finally, a 3-D picture of the solid is shown (with variable scaling).

The procedure for computing the volume of the solid follows.

1. The thickness of a cross-sectional slice is $dx$.
2. The area of this cross-section is
   \[ A(x) = \pi r_o^2 - \pi r_i^2 = \pi \left( (2 - x^4)^2 - 1^2 \right) = \pi \left( 3 - 4x^4 + x^8 \right). \]
3. The volume of the slice is
   \[ dV = A(x) \, dx = \pi \left( 3 - 4x^4 + x^8 \right) \, dx. \]
4. Now add up the volumes of the slices to obtain the volume of the entire solid.
   \[
   \int dV = \int_{-1}^{1} \pi \left( 3 - 4x^4 + x^8 \right) \, dx
   = 2\pi \int_0^1 3 - 4x^4 + x^8 \, dx \quad \text{[via symmetry]}
   = \left. \left( 2\pi \left( 3x - \frac{4}{5}x^5 + \frac{1}{9}x^9 \right) \right) \right|_0^1
   = 2\pi \left( 3 - \frac{4}{5} + \frac{1}{9} \right) - (0) = \frac{208\pi}{45} \approx 14.52
   \]
   The volume is $\frac{208}{45}\pi \approx 14.52$ cm$^3$.

438/12

Find the volume of the solid obtained by rotating the region bounded by the lines $y = x$, $y = 0$, $x = 2$, and $x = 4$, about the line $x = 1$.

Solution

Preliminary work  The planar region lies in the first quadrant. It consists of two subregions: a square surmounted by a triangle. Revolve the region about the line $x = 1$.

The cross-section is a circular washer (ring or annulus) of inner radius $r_i$ and outer radius $r_o$, perpendicular to the line $x = 1$ with center on said line (its axis of symmetry). Finally, a 3-D picture of
the solid is shown (with variable scaling). The solid is bounded by four surface patches: an inner circular cylinder, an outer circular cylinder, a top cone, and a bottom washer.

The procedure for computing the volume of the solid follows.

1. The thickness of a cross-sectional slice is $dy$.
2. The area of this cross-section is

$$A(y) = \pi r_o^2 - \pi r_i^2 = \pi \left( r_o^2 - r_i^2 \right)$$

$$= \left\{ \begin{array}{l} \pi \left( 4 - 1 \right)^2 - \left( x - 1 \right)^2, \quad \text{[triangular region]} \\ \pi \left( 4 - 1 \right)^2 - \left( 2 - 1 \right)^2, \quad \text{[square region]} \\ \pi \left( 9 - (y - 1)^2 \right), \quad \text{[triangular region]} \\ 8\pi, \quad \text{[square region]} \end{array} \right. $$

since $x = y$ is the slanted boundary of the triangular region.

3. The volume of the slice is

$$dV = A(y) dy = \left\{ \begin{array}{l} \pi \left( 9 - (y - 1)^2 \right) dy, \quad \text{[triangular region]} \\ 8\pi dy, \quad \text{[square region]} \end{array} \right. $$

4. Now add up the volumes of the slices to obtain the volume of the entire solid.

$$\int dV = \int_0^2 8\pi dy + \pi \int_2^4 9 - (y - 1)^2 dy$$

$$= (8\pi) (2) + \left( \pi \left( 9y - \frac{1}{3} (y - 1)^3 \right) \right]_2^4$$

$$= 16\pi + 27\pi - 17\frac{7}{3}\pi$$

$$= 25\frac{1}{3}\pi = \frac{76\pi}{3} \approx 79.59$$

The volume is $\frac{76}{3}\pi \approx 79.59$ cm$^3$.

438/24

Find the volume of the solid obtained by rotating the region bounded by the curves $x = y^3$, $y = 2$, and $x = 0$, about the line $x = 8$.

**Solution**

**Preliminary work**  The planar region lies in the first quadrant. Revolve the region about the line $x = 8$.

The cross-section is a circular washer (ring or annulus) of inner radius $r_i = 8 - x$ and outer radius $r_o = 8 - 0 = 8$, perpendicular to the line $x = 8$ with center on said line (its axis of symmetry). Finally, a 3-D picture of the solid is shown (viewed from below).

The procedure for computing the volume of the solid follows.

1. The thickness of a cross-sectional slice is $dy$.
2. The area of this cross-section is

$$A(y) = \pi r_o^2 - \pi r_i^2 = \pi \left( r_o^2 - r_i^2 \right)$$

$$= \pi \left[ (8^2 - (8 - y)^2) = \pi \left( 16y - y^2 \right) \right. $$

$$= \pi \left( 16y^3 - y^6 \right)$$

since $x = y^3$ is the curved boundary of the region.

3. The volume of the slice is

$$dV = A(y) dy = \pi \left( 16y^3 - y^6 \right) dy.$$

4. Now add up the volumes of the slices to obtain the volume of the entire solid.

$$\int dV = \pi \int_0^2 16y^3 - y^6 dy$$

$$= \left( \pi \left( 4y^4 - \frac{1}{7}y^7 \right) \right]_0^2$$

$$= 64 - \frac{128}{7} \pi - 0$$

$$= \frac{320\pi}{7} \approx 143.62$$

The volume is $\frac{320}{7}\pi \approx 143.62$ cm$^3$. 

3
Find the volume of the solid obtained by rotating the region bounded by \( y = \sec x, \ y = 1, \ x = -1, \ \text{and} \ x = 1 \), about the \( x \)-axis.

**Solution**

**Preliminary work**  The planar region lies in the upper half-plane. Revolve it about the \( x \)-axis.

The cross-section is a circular washer (ring or annulus) of inner radius \( r_i = 1 \) and outer radius \( r_o = \sec x \), perpendicular to the \( x \)-axis (its axis of symmetry). Finally, a 3-D picture of the solid is shown.

The procedure for computing the volume of the solid follows.

1. The thickness of a cross-sectional slice is \( dx \).
2. The area of this cross-section is
   \[
   A(x) = \pi r_o^2 - \pi r_i^2 = \pi \left( \sec^2 x - 1 \right).
   \]
3. The volume of the slice is
   \[
   dV = A(x) \, dx = \pi \left( \sec^2 x - 1 \right) \, dx.
   \]
4. Now add up the volumes of the slices to obtain the volume of the entire solid.
   \[
   \int dV = \int_{-1}^{1} \pi \left( \sec^2 x - 1 \right) \, dx = 2 \int_{0}^{1} \pi \left( \sec^2 x - 1 \right) \, dx
   \]
   \[
   = 2\pi \left( \tan x - x \right) \bigg|_{0}^{1}
   \]
   \[
   = 2\pi \left( \tan 1 - 1 \right) \approx 3.50 \text{ cm}^3.
   \]
   The volume is \( 2\pi \left( \tan 1 - 1 \right) \approx 3.50 \text{ cm}^3 \).

Find the volume of the solid \( S \) whose base is the parabolic region
\[
\left\{(x, y) : x^2 \leq y \leq 1, x \geq 0 \right\}
\]
and whose cross-sections perpendicular to the \( x \)-axis are isosceles triangles with height equal to the base.

**Solution**

**Preliminary work**  The base of the solid is in the first quadrant.

The cross-section is a triangular region whose base and height are equal. Finally, a 3-D picture of the solid is shown.

The procedure for computing the volume of the solid follows.

1. The thickness of a cross-sectional slice is \( dx \).
2. The area of this cross-section is
   \[
   A(x) = \frac{1}{2} \cdot \left( 1 - y \right)^2 = \frac{1}{2} \left( 1 - x^2 \right)^2 = \frac{1}{2} x^4 - x^2 + \frac{1}{2}.
   \]
3. The volume of the slice is
   \[
   dV = A(x) \, dx = \left( \frac{1}{2} x^4 - x^2 + \frac{1}{2} \right) dx.
   \]
4. Now add up the volumes of the slices to obtain the volume of the entire solid.
   \[
   \int dV = \int_{0}^{1} \left( \frac{1}{2} x^4 - x^2 + \frac{1}{2} \right) dx
   \]
   \[
   = \left[ \frac{1}{10} x^5 - \frac{1}{3} x^3 + \frac{1}{2} x \right]_{0}^{1}
   \]
   \[
   = \left( \frac{1}{10} - \frac{1}{3} + \frac{1}{2} \right) - 0
   \]
   \[
   = \frac{4}{15} \approx 0.27
   \]
   The volume is \( \frac{4}{15} \approx 0.27 \text{ cm}^3 \). 
A log 10 m long is cut at 1-meter intervals and its cross-sectional areas $A_k$ (at a distance $x$ from the end of the log) are listed in the following table. Use the Midpoint Rule with $n = 5$ to estimate the volume of the log.

<table>
<thead>
<tr>
<th>$x$ (m)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$ (m²)</td>
<td>0.68</td>
<td>0.65</td>
<td>0.64</td>
<td>0.61</td>
<td>0.58</td>
<td>0.59</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$ (m)</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$ (m²)</td>
<td>0.53</td>
<td>0.55</td>
<td>0.52</td>
<td>0.50</td>
<td>0.48</td>
</tr>
</tbody>
</table>

**Solution**

Since $\Delta x_k = \Delta x = 2$, the Midpoint Rule gives

$$V \approx \sum_{k=1}^{5} A(x_k^*) \Delta x_k = \Delta x \sum_{k=1}^{5} A(x_k^*)$$

$$= 2 \left( A(1) + A(3) + A(5) + A(7) + A(9) \right)$$

$$= 2 (0.65 + 0.61 + 0.59 + 0.55 + 0.50)$$

$$= 5.80 \text{ cm}^3.$$

**MATLAB Examples**

As we saw in the hand examples, the difficult part of the problem is the setup. Once we have an integral that represents the volume, the rest is entirely mechanical. To compute the integral by hand we used the Fundamental Theorem of Calculus, which involved antidifferentiation, function evaluation, and subtraction. We may compute an integral in one step using MATLAB’s `int` command or the $\int$ command on a TI-89. To mimick hand work and see our intermediate steps, use the `smi` command in MATLAB or on a TI-89. Here are the integrals from the hand examples computed via machine power.

All graphics depicted in the hand examples were drawn with MATLAB. We saw how to plot curves and do area fills in Section 7.1. This is all that is necessary to produce pictures of the 2-D planar regions and cross-sections involved in problems. (The 3-D pictures involve parametric surfaces and/or curves. These are beyond the scope of this course. We shall encounter them at the end of Math 253—Calc 3.)

**s438x02: Solution**

```matlab
sym x
f = pi*x^3; pretty(f)
```

**s438x11: Solution**

```matlab
% % Stewart 438/11s: Volume integral
% sym x
f = pi * (3 - 4*x^4 + x^8); pretty(f)
V = int(f, x, -1, 1) % Just compute the answer.
V = 208/45*pi
V = smi(f, [x -1 1]); % Show the steps.
STEPWISE (MULTIPLE) INTEGRATION!
Antiderivative w.r.t. x:

$$\frac{4}{8} \pi (3 - 4x^4 + x^8)$$

When $x = 1$:

$$\frac{104}{45} \pi$$

When $x = -1$:

$$-\frac{104}{45} \pi$$

Difference:

$$\frac{208}{45} \pi$$

Answer (above) and approximation (below)

14.5211
%
```

**s438x12: Solution**

```matlab
% % Stewart 438/12s: Volume integral
% sym x
f = pi*x^3; pretty(f)
```
eval(V)
ans =
79.5870

echo off; diary off

s438x24: Solution

% Stewart 438/24s: Volume integral
% Just compute the answer.
V = pi*int(16*y^3 - y^6, y, 0, 2);
pretty(V)

eval(V)
ans =
143.6157

echo off; diary off

s438x29: Solution

% Stewart 438/29s: Volume integral
% Just compute the answer.
V = simple(pi*int(sec(x)^2 - 1, x, -1, 1));
pretty(V)

eval(V)
ans =
3.5023

echo off; diary off

s439x58alt: Solution

% Stewart 439/58alt.s: Volume integral
f = x^4 / 2 - x^2 + 1/2; pretty(f)

V = int(f, x, 0, 1) % Just compute the answer.
V =
4/15
V = simf([x 0 1]); % Show the steps.
STEPWISE (MULTIPLE) INTEGRATION!
Antiderivative w.r.t. x:

\[
\begin{align*}
\int_{0}^{1} f(x) \, dx & = \frac{1}{10} x - \frac{1}{3} x + \frac{1}{2} x \\
\text{When } x & = 1: \\
& = 4/15 \\
\text{When } x & = 0: \\
& = 0 \\
\text{Difference:} & = 4/15
\end{align*}
\]
Answer (above) and approximaton (below)

0.2667