Summary

Volume by cylindrical shells

Imagine a tin can with neither top nor bottom whose axis of symmetry is the line \( L = \{(x, y, z) : x = k, z = 0\} \). This is a cylindrical shell of radius \( r \), height \( h \), and thickness \( dx \). Its volume is \( dV = 2\pi rh \, dx \) since a strip of height \( h \) and thickness \( dx \) is swept around a circumference of \( 2\pi r \). If \( a \leq x \leq b \), then the solid of revolution consisting of the collection of all such cylindrical shells has volume

\[
\int_a^b dV = \int_a^b 2\pi rh \, dx.
\]

Here \( h \) and \( r \) are functions of \( x \).

Analogously, if the axis of symmetry is \( L = \{(x, y, z) : y = k, z = 0\} \), then the solid has volume

\[
\int_a^b dV = \int_a^b 2\pi rh \, dy,
\]

where \( h \) and \( r \) are functions of \( y \).

Cylindrical shells compared with disks/washers

- With shells, the differential area strip \( h \, dx \) or \( h \, dy \) is revolved about an axis of symmetry parallel to the strip.
- With disks/washers, this differential strip is revolved about an axis of symmetry perpendicular to the strip.

Hand Examples

For specificity, we assume lengths are in centimeters.

444/2

Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by \( y = 1/x \), \( y = 0 \), \( x = 1 \), and \( x = 10 \) about the \( y \)-axis.

Solution

Preliminary work The planar region lies in the first octant. Revolve the region about the \( y \)-axis.

444/12

Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by \( y^2 - 6y + x = 0 \) and \( x = 0 \) about the \( x \)-axis.

Solution

Preliminary work The planar region lies in the first octant. Revolve the region about the \( x \)-axis.
Revolve the region about the $x$-axis. Preliminary work The planar region lies in the first octant. Revolve the region about the $x$-axis.

Solution

Preliminary work The planar region lies in the first octant. Revolve the region about the $x$-axis.

The procedure for computing the volume of the solid follows.

1. The thickness of a cylindrical shell is $dy$.
2. The surface area of the outside of a typical cylindrical shell is
   \[
   A(y) = 2\pi rh = 2\pi y(6y^2 - y^2) = \pi(12y^2 - 2y^3)
   \]
3. The volume of the cylindrical shell is
   \[
   dV = A(y)\,dy = \pi(12y^2 - 2y^3)\,dy.
   \]
4. Now add up the volumes of the cylindrical shell to obtain the volume of the entire solid.
   \[
   \int dV = \int_0^6 \pi(12y^2 - 2y^3)\,dy = \pi\left(4y^3 - \frac{e}{2}y^4\right)\bigg|_0^6 = (4(216) - \frac{3}{2}(216))\pi - 0 = 216\pi \approx 678.58
   \]

The volume is $216\pi \approx 678.58$ cm$^3$.

444/20

Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by $y = 4x - x^2$ and $y = 8x - 2x^2$ about the line $x = -2$.

A typical cylindrical shell appears below. It has radius $y$, height $x$, and thickness $dy$. Finally, a 3-D picture of the solid is shown.

The procedure for computing the volume of the solid follows.

1. The thickness of a cylindrical shell is $dx$.
2. The surface area of the outside of a typical cylindrical shell is
   \[
   A(x) = 2\pi rh = 2\pi x(8x - x^2) = \pi(12x^2 - 2x^3)
   \]
3. The volume of the cylindrical shell is
   \[
   dV = A(x)\,dx = \pi(16x + 4x^2 - 2x^3)\,dx
   \]
4. Now add up the volumes of the cylindrical shell to obtain the volume of the entire solid.
   \[
   \int dV = \int_0^4 \pi(16x + 4x^2 - 2x^3)\,dx = \pi\left(8x^3 + \frac{4}{3}x^3 - \frac{2}{1}x^4\right)\bigg|_0^4 = \pi(128 + \frac{256}{3} - 128) = \frac{256\pi}{3} \approx 268.08
   \]

The volume is $\frac{256\pi}{3} \approx 268.08$ cm$^3$.

MATLAB Examples

The first three examples just use MATLAB to recompute the volume integrals from the hand examples. We then compute the volume of a solid that involves several numerical (as opposed to symbolic) computations.
Example A (the “hubcap” example)

Let a region in the first quadrant be bounded by the lines $x = 0$ and $y = 16$ along with the curve

$$x = f(y) = e^{\sin y} + \ln y - \cos (\tan^{-1} y) - 2.$$

Use the method of cylindrical shells to find the volume of the solid obtained by rotating the stated region about the $x$-axis. [NOTE: There are three values of $y$ in the interval $(0, 16)$ for which $f(y) = 0$. Use the largest of these values as the $y$-coordinate of the “southwest” boundary of the region; i.e., $y = c \approx 5.4644$, as shown in the MATLAB code.]

Solution

Preliminary work  The planar region lies in the first octant. Revolve the region about the $x$-axis.

A typical cylindrical shell appears below. It has radius $y$, height $x$, and thickness $dy$. Finally, a 3-D picture of the solid is shown.
The procedure for computing the volume of the solid follows.

1. The thickness of a cylindrical shell is $dy$.

2. The surface area of the outside of a typical cylindrical shell is

   
   \[ A(y) = 2\pi rh = 2\pi yx = 2\pi y \left( e^{\sin y} + \ln y - \cos \left( \tan^{-1} y \right) - 2 \right) . \]

3. The volume of the cylindrical shell is

   \[ dV = A(y) dy = 2\pi y \left( e^{\sin y} + \ln y - \cos \left( \tan^{-1} y \right) - 2 \right) dy. \]

4. Now add up the volumes of the cylindrical shell to obtain the volume of the entire solid.

   \[
   \int dV = \int_{c}^{16} 2\pi y \left( e^{\sin y} + \ln y - \cos \left( \tan^{-1} y \right) - 2 \right) dy \\
   \approx 1253.56
   \]

   The volume is approximately 1253.56 cm$^3$.

We used MATLAB’s `quad` command to compute the volume numerically.

```matlab
% Stewart 7.3/Example A-n: Volume integral
% c = fzero(@f, 5)
c = 5.4644
V = quad(@g, c, 16)
V = 1.2536e+03
fprintf('The volume is approximately %7.2f cm^3.
', V)
The volume is approximately 1253.56 cm^3.

% echo off; diary off
%----------------------
function x = f(y)
x = exp(sin(y)) + log(y) - cos(atan(y)) - 2;
%----------------------
function x = g(y)
x = 2.*pi.*y.*(exp(sin(y)) + log(y) - cos(atan(y)) - 2);
```