**Executive Summary**

(Please see 9.5R, the regular lecture, for the full treatment.)

The point on which a thin flat plate (lamina) balances horizontally is called the **center of mass (CM)** or **center of gravity** of the plate. In this situation mass is distributed continuously. Let $\rho$ be the mass density of the plate. Here $\rho$ may be constant or variable (i.e., depend on $x$ and/or $y$).

$$ \text{mass } m = \iint_D \rho \, dA $$

$$ \text{center of mass } [\bar{x}, \bar{y}] = \frac{1}{m} \iint_D \rho \, [x, y] \, dA $$

The moment with respect to the $x$-axis is $M_x = m \bar{y}$, whereas the moment with respect to the $y$-axis is $M_y = m \bar{x}$. (Recall that the directed distance from a point to the $x$-axis is $y$, whereas said distance to the $y$-axis is $x$.)

You do these problems with machines, folks—either your TI-89 calculator or a computer with MATLAB. (To convince yourself of this, see the hand examples in 9.5R, the regular lecture.)

**TI-89 Examples**

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Find the centroid (center of mass of a flat plate of uniform density) of the region bounded by the curves $y = 1 - x^2$ and $y = 0$.

**Solution**

Here is a diagram showing the region and its center of mass.

First compute the mass

$$ m = \iint_D \rho \, dA = \int_{-1}^{1} \int_{0}^{1-x^2} \rho \, dy \, dx = \frac{4\rho}{3} $$

then compute the center of mass. (Here $\rho$ is constant.)

$$ [\bar{x}, \bar{y}] = \frac{1}{m} \iint_D \rho \, [x, y] \, dA $$

$$ = \frac{1}{4\rho/3} \int_{-1}^{1} \int_{0}^{1-x^2} \rho \, [x, y] \, dy \, dx $$

$$ = \left[ 0, \frac{2}{5} \right] = [0.00, 0.40] $$

These two steps are easy to do on a TI-89, especially if you use the MuInt (multiple integral) menu in the TAMUCALC package. (You’ll find $\rho$ on the Calc menu.)

- $\int (\int (\rho, y, 0, 1-x\wedge2), x, -1, 1) \rightarrow \mathbf{m}$
- $\int (\int (\rho \ast [x, y], y, 0, 1-x\wedge2), x, -1, 1)/\mathbf{m}$

The second command is even easier than you think. Just change the $\rightarrow$ (from the store operation) in the entry line to $/$ for division and post-multiply $\rho$ by $[x, y]$.

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Find the centroid (center of mass of a flat plate of uniform density) of the region bounded by the curves $y = \sin x$, $y = 0$, $x = 0$, and $x = \frac{\pi}{2}$.

**Solution**

Here is a diagram showing the region and its center of mass.

First compute the mass

$$ m = \iint_D \rho \, dA = \int_{0}^{\pi/2} \int_{0}^{\sin x} \rho \, dy \, dx = \rho $$

then compute the center of mass. (Here $\rho$ is constant.)

$$ [\bar{x}, \bar{y}] = \frac{1}{m} \iint_D \rho \, [x, y] \, dA $$

$$ = \frac{1}{\rho} \int_{0}^{\pi/2} \int_{0}^{\sin x} \rho \, [x, y] \, dy \, dx $$

$$ = \left[ 1, \frac{\pi}{8} \right] \approx [1.00, 0.39] $$

- $\int (\int (\rho, y, 0, \sin (x)), x, 0, \pi/2) \rightarrow \mathbf{m}$
- $\int (\int (\rho \ast [x, y], y, 0, \sin (x)), x, 0, \pi/2)/\mathbf{m}$
Find the mass and center of mass of the lamina (flat plate) that occupies the triangular region $D$ in the $xy$-plane with vertices $(0, 0), (1, 1),$ and $(4, 0)$ and has variable density $\rho = x$.

Solution

Here is a plot showing the region and its center of mass (+).

With our general formulation, the fact that the density is variable presents no difficulty whatsoever! First compute the mass

$$m = \iint_D \rho \, dA = \int_0^1 \int_y^{4-3y} x \, dx \, dy = \frac{10}{3}$$

then compute the center of mass.

$$[\bar{x}, \bar{y}] = \frac{1}{m} \iint_D \rho \, [x, y] \, dA$$

$$= \frac{1}{10/3} \int_0^1 \int_y^{4-3y} x[x, y] \, dx \, dy$$

$$= \left[ \frac{21}{10} \frac{3}{10} \right] = [2.1, 0.3]$$

- $x \rightarrow \rho$
- $\int (\int \rho, x, y, 4-3y, y, 0, 1) \rightarrow m$
- $\int (\int (\rho \star [x, y], x, y, 4-3y), y, 0, 1)/m$
- DelVar $\rho$  (Note that NewProb clears only single letter Roman identifiers—not Greek ones!)

MATLAB Examples

We’ll repeat the TI-89 examples using MATLAB. The syntax for the integrals involved is almost identical! Just replace $\int$ with `int`. For brevity, we’ll type `p` for the density instead of `rho` (for $\rho$).

s560x06 [560/6 revisited]

Find the centroid (center of mass of a flat plate of uniform density) of the region bounded by the curves $y = 1 - x^2$ and $y = 0$. 

Solution

Here is a diagram showing the region and its center of mass.

Find the centroid (center of mass of a flat plate of uniform density) of the region bounded by the curves $y = \sin x$, $y = 0$, $x = 0$, and $x = \frac{\pi}{2}$.

Solution

Here is a diagram showing the region and its center of mass.
Find the mass and center of mass of the lamina (flat plate) that occupies the triangular region $D$ in the $xy$-plane with vertices $(0, 0), (1, 1),$ and $(4, 0)$ and has variable density $\rho = x$.

**Solution**

Here is a plot showing the region and its center of mass (+).

![Plot of the triangular region and its center of mass](image)

The center of mass is

$$(\bar{x}, \bar{y}) = \left( \frac{8}{3 + 6 \ln 2}, \frac{5}{6 + 12 \ln 2} \right) \approx (1.1175, 0.3492)$$
(a - b) x
y = --------- + b
h

% Median lines
M1 = cline2pt(V1, (V2+V3)/2); pretty(M1)

\[
x (a + b)
y = \frac{---------}{h}
\]

M2 = cline2pt(V2, (V1+V3)/2); pretty(M2)

\[
2 a x - x b + b h
y = \frac{1/2 \frac{---------}{h}}{}
\]

M3 = cline2pt(V3, (V1+V2)/2); pretty(M3)

\[
(-2 b + a) x
y = \frac{---------}{h} + b
\]

% Common intersection of medians
I1 = solve(M1, M2, x, y);
I2 = solve(M1, M3, x, y);
I3 = solve(M2, M3, x, y);
I = [I1.x I1.y; I2.x I2.y; I3.x I3.y]; pretty(I)

\[
[1/3 h 1/3 a + 1/3 b]
[1/3 h 1/3 a + 1/3 b]
[1/3 h 1/3 a + 1/3 b]
\]

m = int(int(p, y, a*x/h, (a-b)*x/h + b), x, 0, h);
pretty(m)

\[
/a - b \\
\frac{1/2 p \frac{---------}{h} + p b h}{
\}
\]

CM = 1/m * int(int(p*[x y], y, a*x/h, (a-b)*x/h + b), x, 0, h);
CM = simple(CM);
pretty(CM) % Centroid is at intersection of medians.

\[
[1/3 h 1/3 a + 1/3 b]
\]

% echo off; diary off